Warm-Up Problems and Lecture Problems
Solutions
April 2, 2003

Series Toolbox:

I. If a_n is a geometric sequence, then $\sum_{n=1}^{\infty} a_n$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. If $|r| < 1$, then $\sum_{n=1}^{\infty} a_n = \frac{a}{1-r}$ where a is the first term and r is the common ratio.

II. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

III. If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ might diverge or it might converge (thing about $\lim_{n\to\infty} \frac{1}{n}$).

1. Determine of the following series diverge or converge. If the series converges, try to find the sum.

   (a) $\sum_{n=1}^{\infty} 5^{-n}$
   Solution: This is a geometric series with $r = \frac{1}{5}$ and first term $a = \frac{1}{5}$. Therefore the sum is: $\frac{(1/5)}{1-(1/5)} = \frac{1}{4}$.

   (b) $\sum_{n=0}^{\infty} 5^{-n}$
   Solution: This is a geometric series with $r = \frac{1}{5}$ and first term $a = 1$. Therefore the sum is $\frac{1}{1-(1/5)} = \frac{5}{4}$.

   (c) $\sum_{n=1}^{\infty} \sqrt[n]{2}$
   Solution: This diverges because $\lim_{n\to\infty} \sqrt[n]{2} = 1 \neq 0$.

   (d) $\sum_{n=1}^{\infty} \frac{1+2^n}{5^n}$
   Solution: This is really two geometric series:

   $$\sum_{n=1}^{\infty} \frac{1+2^n}{5^n} = \sum_{n=1}^{\infty} \frac{1}{5^n} + \sum_{n=1}^{\infty} \frac{2^n}{5^n}$$

   The first series has $r = 1/5$ and $a = 1/5$. The second series has $r = 2/5$ and $a = 2/5$. Therefore the sum is equal to

   $$\sum_{n=1}^{\infty} \frac{1+2^n}{5^n} = \frac{1/5}{1-(1/5)} + \frac{2/5}{1-(2/5)} = \frac{1}{4} + \frac{2}{3} = \frac{11}{12}$$

   (e) $\sum_{n=0}^{\infty} \sin^n 1$
   Solution: This is geometric with $r = \sin 1$ and $a = 1$. Since $\sin 1 \approx 0.8415 < 1$, the series converges and is equal to:

   $$\frac{1}{1-\sin 1} \approx 6.308$$

2. (a) Suppose you know that $\sum_{n=1}^{\infty} a_n$ converges. How about $\sum_{n=100}^{\infty} a_n$? Why or why not?
   Solution: $\sum_{n=100}^{\infty} a_n$ will also converge. Here is how you would show this in terms of partial sums. Let s_N be the partial sum for $\sum_{n=1}^{\infty} a_n$ and let q_N be the partial sum for $\sum_{n=100}^{\infty} a_n$:

   $$s_N = \sum_{n=1}^{N} a_n \qquad q_N = \sum_{n=100}^{N} a_n$$

Also, let $A = \sum_{n=1}^{99} a_n$. Then, it is easy to see that when $N > 100$, $q_N = s_N - A$. Therefore, $\lim_{N\to\infty} q_N = \lim_{N\to infty} s_N - A$ which converges.

(b) Suppose you know that $\sum_{n=1}^{\infty} a_n$ diverges. How about $\sum_{n=100}^{\infty} a_n$? Why or why not?
Solution: It will also diverge. The point to both of these problems is that what happens at the beginning of the sequence will not influence if the series diverges or converges, it will only influence what the series converges to (if it converges).

Lecture Problems

3. Use the integral test to determine if the following series converge or diverge

   (a) $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$
   Solution:

   $$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t \to infty} \arctan x\big|_1^t = \frac{\pi}{4}$$

   So, the series converges.

   (b) $\sum_{n=1}^{\infty} \frac{4}{4n-1}$
   Solution:

   $$\int_1^{\infty} \frac{4}{4x-1}\,dx = \lim_{t \to \infty} \ln(4x-1)\big|_1^t = \infty$$

   So the series diverges.

4. Use the comparison test to determine if the following series converge or diverge

   (a) $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$
   Solution:

   $$\frac{1}{k^2+1} \leq \frac{1}{k^2}$$

   and the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges so therefore $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$ also converges.

   (b) $\sum_{n=1}^{\infty} \frac{4}{4n-1}$
   Solution:

   $$\frac{4}{4n-1} \geq \frac{4}{4n} = \frac{1}{n}$$

   and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so therefore $\sum_{n=1}^{\infty} \frac{4}{4n-1}$ also diverges.