Warm-Up Problems and Lecture Problems
Solutions
April 18, 2003

1. (a) Find the derivative of $\frac{1}{1-x}$.
   Solution: $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$.

   (b) Find a power series for $\frac{1}{(1-x)^2}$.
   Solution: Since we know the power series for $\frac{1}{1-x}$, we can take the derivative to get a power series for $\frac{1}{(1-x)^2}$.

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\
&= \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) \\
&= 1 + 2x + 3x^2 + 4x^3 + \cdots \\
&= \sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=1}^{\infty} nx^{n-1}
\end{aligned}$$

2. Find a power series for e^{2x}.
   Solution: To find this power series, we can substitute $2x$ into the power series for e^x:

$$\begin{aligned}
e^{2x} &= \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n x^n}{n!} \\
&= 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots
\end{aligned}$$

3. Find a power series, centered at $x = 0$, for $\frac{1}{4+x}$
Solution:

$$\begin{aligned}
\frac{1}{4+x} &= \frac{1}{4(1+x/4)} = \left(\frac{1}{4}\right)\frac{1}{1+(x/4)} \\
&= \left(\frac{1}{4}\right)\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n = \left(\frac{1}{4}\right)\sum_{n=0}^{\infty}\frac{x^n}{4^n} = \sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}} \\
&= \frac{1}{4} + \frac{x}{4^2} + \frac{x^2}{4^3} + \frac{x^3}{4^4} + \cdots
\end{aligned}$$

4. Find a power series, centered at $x = 0$, for $\ln(4 + x)$.
Solution:

$$\begin{aligned}
\ln(4+x) &= \int \frac{1}{4+x}\,dx \\
&= \int \left(\frac{1}{4} + \frac{x}{4^2} + \frac{x^2}{4^3} + \frac{x^3}{4^4} + \cdots\right)\,dx \\
&= C + \frac{x}{4} + \frac{x^2}{2\cdot 4^2} + \frac{x^3}{3\cdot 4^3} + \frac{x^4}{4\cdot 4^4} + \cdots
\end{aligned}$$

Plug in $x = 0$ to get $C = \ln 4$:

$$\ln(4+x) = \ln(4) + \frac{x}{4} + \frac{x^2}{2\cdot 4^2} + \frac{x^3}{3\cdot 4^3} + \frac{x^4}{4\cdot 4^4} + \cdots$$

Lecture Problems

5. Find the Taylor series for $\sin x$, centered at $x = \frac{\pi}{6}$.

Solution: You can make a chart as in class or use the formula $c_n = \frac{f^{(n)}(a)}{n!}$. In any case, you should get:

$$\begin{aligned}\sin(x) &= \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) - \frac{1}{2}\frac{(x-\frac{\pi}{6})^2}{2!} - \frac{\sqrt{3}}{2}\frac{(x-\frac{\pi}{6})^3}{3!} \\ &\quad + \frac{1}{2}\frac{(x-\frac{\pi}{6})^4}{4!} + \frac{\sqrt{3}}{2}\frac{(x-\frac{\pi}{5})^3}{5!} - \frac{1}{2}\frac{(x-\frac{\pi}{6})^6}{6!} - \frac{\sqrt{3}}{2}\frac{(x-\frac{\pi}{6})^7}{7!} + \cdots\end{aligned}$$