Warm-Up Problems and Lecture Problems

Solutions

April 11, 2003

1. Determine if the following series converge or diverge. If the series converges, determine if the series converges conditionally or if the series converges absolutely.

   (a) $\sum_{n=2}^{\infty}(-1)^n \ln n$
   Solution: diverges: $\lim_{n\to\infty} a_n \neq 0$.

   (b) $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}$
   Solution: converges by alternating series test. But, this series converges conditionally since the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the comparison test ($\frac{1}{n} < \frac{1}{\ln n}$).

   (c) $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n+2}{n^3}$
   Solution: converges absolutely by the limit comparison test. Compare $\sum_{n=1}^{\infty} \frac{n+2}{n^3}$ to $\sum_{n=1}^{\infty} \frac{n}{n^3}$. (Since the series of absolute values converges, we know that the alternating series also converges).

   (d) $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n+2}{n^2}$
   Solution: This certainly does not converge absolutely (use a limit comparison test, compare to $\sum_{n=1}^{\infty} \frac{n}{n^2}$). But, it does converge conditionally. $\lim_{n\to\infty} b_n = 0$. And, if you let $f(x) = \frac{x+2}{x^2}$, then $f'(x) = -\frac{(x+4)}{x^3} < 0$ and therefore the terms $\frac{n+2}{n^2}$ are decreasing.

Lecture and homework problems

2. Using the ratio test, determine the radius of convergence and the interval of convergence for the following series:

   (a) $\sum_{n=0}^{\infty} 3^n x^n$
   Solution: Letting $a_n = 3^n x^n$:

   $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1} x^{n+1}}{3^n x^n} \right| = |3x|$$

   Since we want this to be less then one, we have $|3x| < 1$ or $|x| < \frac{1}{3}$. $R = \frac{1}{3}$. To find the interval of convergence, you need to test convergence (by hand) when $x = \frac{1}{3}$ and $x = -\frac{1}{3}$. If $x = \frac{1}{3}$ then the series is $\sum_{n=0}^{\infty} 1$, which diverges. If $x = -\frac{1}{3}$ then the series is $\sum_{n=0}^{\infty} (-1)^n$ which diverges. So, the interval of convergence is $(-\frac{1}{3}, \frac{1}{3})$.

   (b) $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$
   Solution: $R = 3$ and the interval of convergence is $(-3, 3)$.

   (c) $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$
   Solution: $R = 3$ and the interval of convergence is $[-3, 3)$.

   (d) $\sum_{n=0}^{\infty} \frac{nx^n}{3^n}$
   Solution: $R = 3$ and the interval of convergence is $(-3, 3)$.