- 25 multiple choice questions worth 4 points each.
- No graphing calculators!
  Any non-graphing, non-differentiating, non-integrating scientific calculator is fine.
- For the multiple choice questions, mark your answer on the answer card.

$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$	$\sin(2A) = 2 \sin A \cos A$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\cos(2A) = \cos^2 A - \sin^2 A$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\tan(2A) = \dfrac{2 \tan A}{1 - \tan^2 A}$
$\sin^2(A/2) = \dfrac{1 - \cos A}{2}$	$\cos^2(A/2) = \dfrac{1 + \cos A}{2}$
$\tan(A/2) = \dfrac{1 - \cos A}{\sin A} = \dfrac{\sin A}{1 + \cos A}$	$\log_a x = \dfrac{\log_b x}{\log_b a}$
$\sin A \sin B = \dfrac{1}{2}[\cos(A - B) - \cos(A + B)]$	$\cos A \cos B = \dfrac{1}{2}[\cos(A - B) + \cos(A + B)]$
$\sin A \cos B = \dfrac{1}{2}[\sin(A + B) + \cos(A - B)]$	$\cos A \sin B = \dfrac{1}{2}[\sin(A + B) - \cos(A - B)]$
$\sin A + \sin B = 2 \sin\left(\dfrac{A+B}{2}\right) \cos\left(\dfrac{A-B}{2}\right)$	$\sin A - \sin B = 2 \cos\left(\dfrac{A+B}{2}\right) \sin\left(\dfrac{A-B}{2}\right)$
$\cos A + \cos B = 2 \cos\left(\dfrac{A+B}{2}\right) \cos\left(\dfrac{A-B}{2}\right)$	$\cos A - \cos B = -2 \sin\left(\dfrac{A+B}{2}\right) \sin\left(\dfrac{A-B}{2}\right)$
$\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$	$\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1 - x^2}}$
$\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1 + x^2}$	$\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1 + x^2}$
$\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{\lvert x \rvert \sqrt{x^2 - 1}}$	$\dfrac{d}{dx}(\csc^{-1} x) = -\dfrac{1}{\lvert x \rvert \sqrt{x^2 - 1}}$
$\displaystyle\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$	$\displaystyle\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$
$\displaystyle\sum_{k=1}^{n} k^3 = \left[\dfrac{n(n+1)}{2}\right]^2$	$\displaystyle\sum_{k=1}^{n} k^3 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

1. Find

$$\lim_{x\to\infty}\left(\frac{x-3}{x+1}\right)^{x/2}$$

(a) $-\infty$, ∞, or DNE (Limit does not exist)
(b) e^{-4}
(c) e^{-2} $\longrightarrow$ **CORRECT**
(d) e^{-1}
(e) 1
(f) 2
(g) e
(h) e^2
(i) e^3
(j) e^4

Solution: This is indeterminant of the form 1^∞.

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x-3}{x+1}\right)^{x/2} &= \lim_{x\to\infty}\exp\left[\ln\left(\frac{x-3}{x+1}\right)^{x/2}\right] = \exp\left[\lim_{x\to\infty}\ln\left(\frac{x-3}{x+1}\right)^{x/2}\right]\\ &=\exp\left[\lim_{x\to\infty}\frac{x}{2}\cdot\ln\left(\frac{x-3}{x+1}\right)\right] = \exp\left[\lim_{x\to\infty}\frac{\ln\left(\frac{x-3}{x+1}\right)}{2/x}\right]\\ &\overset{LH}{=}\exp\left[\lim_{x\to\infty}\frac{\frac{4}{(x-3)(x+1)}}{-2/x^2}\right] = \exp\left[\lim_{x\to\infty}\frac{-4x^2}{2(x-3)(x+1)}\right] = \exp(-2) = e^{-2}\end{aligned}$$

2. Find

$$\lim_{x\to 4}\left(\frac{x-3}{x+1}\right)^{x/2}$$

(a) $-\infty$, ∞, or DNE (Limit does not exist)
(b) $1/25$ $\longrightarrow$ **CORRECT**
(c) $1/5$
(d) e^{-2}
(e) e^{-1}
(f) 1
(g) 2

(h) e
(i) 5
(j) e^2
(k) 25

Solution: This is not indeterminant

$$\lim_{x\to 4}\left(\frac{x-3}{x+1}\right)^{x/2} = \left(\frac{4-3}{4+1}\right)^{4/2} = \frac{1}{25}$$

3. Find

$$\lim_{x\to\infty}\frac{\ln(100x)}{x^{0.01}}$$

(a) 0 $\longrightarrow$ **CORRECT**
(b) 0.01
(c) 100
(d) ∞
(e) DNE, the limit does not exist
(f) Something else

Solution:

$$\lim_{x\to\infty}\frac{\ln(100x)}{x^{0.01}} \overset{LH}{=} \lim_{x\to\infty}\frac{1/x}{0.01x^{-0.99}} = \lim_{x\to\infty}\frac{100}{x^{0.01}} = 0$$

4. Find

$$\lim_{x\to\infty}\frac{x^{100}}{e^{0.01x}}$$

(a) 0 $\longrightarrow$ **CORRECT**
(b) 0.01
(c) 100
(d) ∞
(e) DNE, the limit does not exist
(f) Something else

Solution: You can use L'Hopital a lot of times:

$$\lim_{x\to\infty}\frac{x^{100}}{e^{0.01x}} \overset{LH}{=} \lim_{x\to\infty}\frac{100x^{99}}{0.01e^{0.01x}} \overset{LH}{=} \cdots \overset{LH}{=} \lim_{x\to\infty}\frac{100\cdot 99\cdot 98\cdots 2\cdot 1}{(0.01)^{100}e^{0.01x}} = 0$$

5. Find

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$$

(a) DNE (Limit does not exist)
(b) $-\infty$
(c) -1
(d) $-1/2$
(e) 0
(f) $1/2 \longrightarrow$ **CORRECT**
(g) 1
(h) ∞

Solution: Check to see that it is a $0/0$ indeterminate form and use L'Hopital's Rule:

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2} \overset{LH}{=} \lim_{x\to 0} \frac{e^x - 1}{2x} \overset{LH}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

6. The graph of $f(x)$ passes through the point $(-1, 4)$. The slope of the line tangent to the graph at the point $(x, f(x))$ is $-2x - 3$. Find $f(0)$.

(a) -3
(b) -2
(c) -1
(d) 0
(e) $2 \longrightarrow$ **CORRECT**
(f) 3
(g) 4
(h) 10

Solution: The slope of the tangent line is $-2x - 3$ means $f'(x) = -2x - 3$. Thus, $f(x) = -x^2 - 3x + C$ for some C. Plugging in the poing $(-1, 4)$ gives $C = 2$ and thus $f(x) = -x^2 - 3x + 2$ and $f(0) = 2$.

7. Find the x-coordinate of the point on the graph of $y = x^2$ that is closest to the point $\left(16, \frac{1}{2}\right)$.

(a) 0
(b) $1/2$
(c) 1

(d) 3/2
(e) 2 $\longrightarrow$ **CORRECT**
(f) 5/2
(g) 3
(h) 7/2
(i) 4
(j) 16
(k) 32

Solution: The distance between the point on the curve (x, x^2) and the point in question is

$$D = \sqrt{(x-16)^2 + (x^2 - 1/2)^2}$$

And the distance squared is $f(x) = (x-16)^2 + (x^2 - 1/2)^2$. Find the derivative, set it equal to 0, solve:

$$f'(x) = 2(x-16) + 2(x^2 - 1/2)(2x) = 4x^3 - 32$$

Solving $f'(x) = 0$ gives $x = 2$. You can test this critical point with the first derivative test to see that it is a minimum.

8. Let $f(t) = \dfrac{t^2 - 1}{t + 3}$. Find $f'(1)$

(a) -1
(b) $-1/9$
(c) 0
(d) 1/9
(e) 1/4
(f) 1/2 $\longrightarrow$ **CORRECT**
(g) 1
(h) 3/2

Solution: Quotient rule

$$f'(t) = \frac{2t(t+3) - (t^2-1)(1)}{(t+3)^2}$$
$$f'(1) = \frac{2(1+3) - (1-1)(1)}{(1+3)^2} = \frac{1}{2}$$

9. Suppose y is a function of x and

$$\sin y + \cos x = \frac{1}{2}$$

Find the slope of the tangent line at the point $\left(\frac{\pi}{3}, 0\right)$

(a) $-\sqrt{3}$
(b) $-\sqrt{3}/2$
(c) $-1/2$
(d) 0
(e) $1/2$
(f) $\sqrt{3}/2 \longrightarrow$ **CORRECT**
(g) $\sqrt{3}$

Solution: Take the derivative implicity

$$\begin{aligned}(\cos y)y' - \sin x &= 0\\ y' &= \frac{\sin x}{\cos y}\\ y'\left(\frac{\pi}{3}, 0\right) &= \frac{\sin\frac{\pi}{3}}{\cos 0} = \frac{\sqrt{3}}{2}\end{aligned}$$

10. Let $L(x)$ be the linearization of $f(x) = \sin x + 1$ at the point $x = \pi$. Find $L(\pi/2)$.

(a) -1
(b) 0
(c) $\pi/2 - 1$
(d) 1
(e) $\pi/2$
(f) $\pi/2 + 1 \longrightarrow$ **CORRECT**
(g) $\pi + 1$

Solution:

$$\begin{aligned}L(x) =& f(\pi) + f'(\pi)(x - \pi) = 1 + (\cos\pi)(x - \pi) = -x + 1 + \pi\\ L(\pi/2) =& -\pi/2 + 1 + \pi = \pi/2 + 1\end{aligned}$$

11. The side of square is increasing at a rate of 3 inches per minute. Find the rate of change of the area of the square, in square inches per minute, when the side length is 2 inches.

(a) 1
(b) 2
(c) 3
(d) 4
(e) 6
(f) 8
(g) 10
(h) 12 $\longrightarrow$ **CORRECT**

Solution: Let x be the side length.

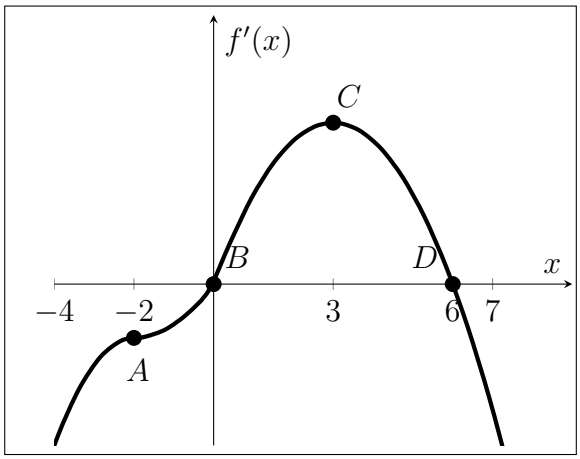
$$A = x^2$$
$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$
$$\left.\frac{dA}{dt}\right|_{x=2} = 2 \cdot 2 \cdot 3 = 12$$

12. Let $f(x) = 2x^3 - 3x^2 - 12x + 2$.
Let m be the minimum of f on the interval $[-3, 1]$.
Let M be the maximum of f on the interval $[-3, 1]$.
Find $M - m$.

(a) -43
(b) -18
(c) -11
(d) 9
(e) 32
(f) 43
(g) 52 $\longrightarrow$ **CORRECT**
(h) f does not have a maximum and/or a minimum.

Solution: $f'(x) = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$
Critical Points: $x = -1, 2$.
Max of 9 at $x = -1$.
Min of -43 at $x = -3$.
$M + m = 52$.

The following will be used for Questions 13-15.
Note, this is a graph of $f'(x)$.
Points $A = (-2, -1)$, $B = (0, 0)$, $C = (3, 3)$ and $D = (6, 0)$ are labeled.
Determine the x-value of all local maxima, local minima and inflection points.



Answers for Questions 13-15:

(a) $x = -2$ (Point A)
(b) $x = 0$ (Point B)
(c) $x = 3$ (Point C)
(d) $x = 6$ (Point D)
(e) Some other point not labeled
(f) More than one point
(g) No points
(h) Not enough information

13. Where does $f(x)$ have a local maxima on the interval $[-4, 7]$?

    Solution: $\longrightarrow$ **CORRECT D**

    Local maxima occur when f' changes from positive to negative. This happens at D.

14. Where does $f(x)$ have a local minimum on the interval $[-4, 7]$?

    Solution: $\longrightarrow$ **CORRECT B**

    Local maxima occur when f' changes from negative to positive. This happens at B.

15. Where does $f(x)$ have an inflection point on the interval $[-4, 7]$?

    Solution: $\longrightarrow$ **CORRECT C**

    Inflection points occur when the sign of f'' changes. We can read this off the graph of f' as the slope of f'. These slopes change from negative to positive at C. Note that the sign of the slope does not change at point A.

16. A particle is moving along the x-axis with position $s(t)$ at time t.

    Suppose you know the following about the position function:

    - $s(-3) = -4$
    - $s'(-3) = -5$
    - $s''(-3) = 12$
    - $s^{(3)}(-3) = -43$ (this is the third derivative of $s(t)$)

    Pick the statement(s) that is true for the particle when $t = -3$.

(a) The particle is not moving
(b) The particle is moving to the right, and speeding up
(c) The particle is moving to the right, and slowing down
(d) The particle is moving to the left, and speeding up
(e) The particle is moving to the left, and slowing down $\longrightarrow$ **CORRECT**
(f) More than one of these is true.
(g) None of these are true.
(h) The situation described is mathematically impossible.

Solution: Since $s'(-3) < 0$, the particle is moving to the left. Since $s''(-3) > 0$ the particle is accelerating to the right. Thus, the particle is slowing down.

Note that $s(-3)$ and $s^{(3)}(-3)$ are irrelevant to what is being asked in this question.

17. Let $F(x)$ be the antiderivative of $f(x) = 2x^3 - 5x^4$ such that $F(0) = 1$. Find $F(1)$.

(a) -1
(b) $-1/2$
(c) 0
(d) $1/2$ $\longrightarrow$ **CORRECT**
(e) 1
(f) $3/2$
(g) 2
(h) $5/2$
(i) 3
(j) $7/2$

Solution: $F(x) = \int f(x)\,dx = -\frac{1}{2}x^4 - x^5 + C$. Since $F(0) = 1$, we must have $C = 1$ and $F(1) = \frac{1}{2} - 1 + 1 = \frac{1}{2}$.

18. Find $\displaystyle\sum_{k=-2}^{3} k^2$

(a) -2
(b) 0
(c) 4
(d) 9

(e) 19 $\longrightarrow$ **CORRECT**

(f) 20

(g) 100

(h) ∞

Solution: $\displaystyle\sum_{k=-2}^{3} k^2 = (-2)^2 + (-1)^2 + (0)^2 + (1)^2 + (2)^2 + (3)^2 = 19$

19. Find the limit of Riemann sums that is equal to the definite integral

$$\int_1^3 \sqrt{x}\, dx$$

(a) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{\frac{k}{n}}$

(b) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n}\sqrt{\frac{k}{n}}$

(c) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{\frac{2k}{n}}$

(d) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n}\sqrt{\frac{2k}{n}}$

(e) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{1+\frac{k}{n}}$

(f) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n}\sqrt{1+\frac{k}{n}}$

(g) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{1+\frac{2k}{n}}$

(h) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n}\sqrt{1+\frac{2k}{n}}$ $\longrightarrow$ **CORRECT**

Solution: We have

$$\int_1^3 \sqrt{x}\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

We must have $\Delta x = \frac{b-a}{n} = \frac{2}{n}$. And, $x_k = a + k\Delta x = 1 + \frac{2k}{n}$. If we let $x_k^* = x_k$ we have $f(x_k) = \sqrt{1+\frac{2k}{n}}$. Putting all this together gives the answer.

20. Find the integral equal to the limit of Riemann sums

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{4}{n} \sin\left(-2 + \frac{4k}{n}\right)$$

(a) $\displaystyle\int_{-2}^{2} \sin x \, dx \longrightarrow$ **CORRECT**

(b) $\displaystyle\int_{-2}^{0} \sin x \, dx$

(c) $\displaystyle\int_{0}^{2} \sin x \, dx$

(d) $\displaystyle\int_{0}^{4} \sin x \, dx$

(e) $\displaystyle\int_{-2}^{2} \sin 4x \, dx$

(f) $\displaystyle\int_{0}^{2} \sin 4x \, dx$

(g) $\displaystyle\int_{0}^{4} \sin 4x \, dx$

21. Approximate the definite integral below using a left hand sum with $n = 3$ (3 subdivisions).

$$\int_{-1}^{5} x^2 \, dx$$

(a) -1

(b) 1

(c) 9

(d) 22 $\longrightarrow$ **CORRECT**

(e) 25

(f) 40

(g) 42

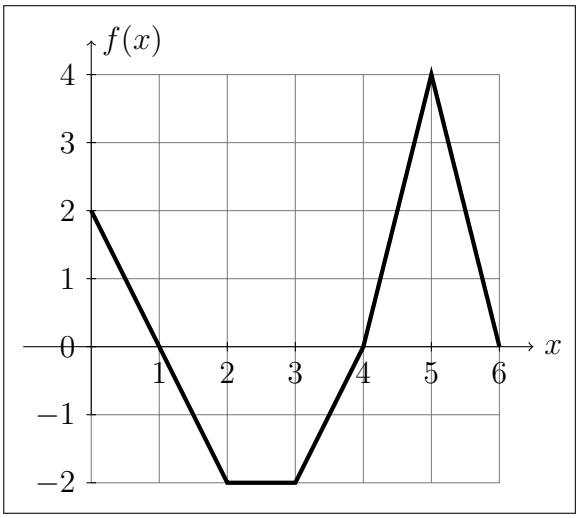
(h) 46

Solution: Subdivide the interval $[-1, 5]$ into 3 subintervals. This gives subdivisions $\{-1, 1, 3, 5\}$, with $\Delta x = 2$. For a left hand sum, we use the left end points to get the heights:

$$\begin{aligned}\text{LHS} =& f(-1)\Delta x + f(1)\Delta x + f(3)\Delta x \\ =& (-1)^2 \cdot 2 + (1)^2 \cdot 2 + 3^2 \cdot 2 = 22\end{aligned}$$

22. Given the graph of $f(x)$ shown below, find

$$\int_1^5 f(x)\,dx$$



Answers for Questions 22

(a) -2 $\longrightarrow$ **CORRECT**

(b) -1

(c) 0

(d) 1

(e) 2

(f) 3

(g) 4

(h) 5

(i) 6

Solution: Add up the area between the curve and the x-axis from $x = 1$ to $x = 5$, making sure to count the area under the x-axis as negative.

$$\begin{aligned}\int_1^5 f(x)\,dx &= \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx \\ &=(-1) + (-2) + (-1) + (2) = -2\end{aligned}$$

23. Suppose $\int_1^8 f(x)\,dx = 12$, $\int_5^1 f(x)\,dx = 3$, $\int_7^8 f(x)\,dx = 4$.

Find $\int_5^7 (2f(x) - 1)\ dx$

(a) 5

(b) 8

(c) 10

(d) 11

(e) 12

(f) 20 $\longrightarrow$ **CORRECT**

(g) 22

Solution: The information given says that

$$\begin{aligned}\int_1^8 f(x)\,dx &= \int_1^5 f(x)\,dx + \int_5^7 f(x)\,dx + \int_7^8 f(x)\,dx \\ 12 &= -3 + \int_5^7 f(x)\,dx + 4 \\ \int_5^7 f(x)\,dx &= 11 \\ \int_5^7 (2f(x)-1)\,dx &= 2\int_5^7 f(x)\,dx - \int_5^7 dx = 22 - 2 = 20\end{aligned}$$

24. Find $\int_{-2}^{2} \sqrt{4-x^2}\,dx$

    (a) 0
    (b) $\frac{1}{2}\pi$
    (c) π
    (d) $\frac{3}{2}\pi$
    (e) $2\pi \longrightarrow$ **CORRECT**
    (f) $\frac{5}{2}\pi$
    (g) 3π
    (h) $\frac{7}{2}\pi$
    (i) 4π

Solution: This is the area of a half circle with radius 2, thus the area is $A = \frac{1}{2}\pi(2)^2 = 2\pi$

25. Find $\int_1^3 (4x^3 - 2x + 1)\,dx$

    (a) 0
    (b) 1
    (c) 3
    (d) $74 \longrightarrow$ **CORRECT**
    (e) 75
    (f) 100

(g) 103

Solution: Use the Fundamental Theorem of Calculus:

$$\int_1^3 (4x^3 - 2x + 1)\, dx = (x^4 - x^2 + x)\Big|_1^3 = 75 - 1 = 74$$