This exam has:

- 18 multiple choice questions worth 4 points each.
- 2 hand graded questions worth 14 points each.

Important:

- No graphing calculators!
  Any non-graphing, non-differentiating, non-integrating scientific calculator is fine.
- For the multiple choice questions, mark your answer on the answer card.
- For the written problems:
  Show all your work for the written problems.
  Your ability to make your solution clear will be part of the grade.
  Use the back of this sheet, if necessary.

$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$	$\sin(2A) = 2 \sin A \cos A$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\cos(2A) = \cos^2 A - \sin^2 A$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\tan(2A) = \dfrac{2 \tan A}{1 - \tan^2 A}$
$\sin^2(A/2) = \dfrac{1 - \cos A}{2}$	$\cos^2(A/2) = \dfrac{1 + \cos A}{2}$
$\tan(A/2) = \dfrac{1 - \cos A}{\sin A} = \dfrac{\sin A}{1 + \cos A}$	$\log_a x = \dfrac{\log_b x}{\log_b a}$
$\sin A \sin B = \dfrac{1}{2}[\cos(A - B) - \cos(A + B)]$	$\cos A \cos B = \dfrac{1}{2}[\cos(A - B) + \cos(A + B)]$
$\sin A \cos B = \dfrac{1}{2}[\sin(A + B) + \cos(A - B)]$	$\cos A \sin B = \dfrac{1}{2}[\sin(A + B) - \cos(A - B)]$
$\sin A + \sin B = 2 \sin\left(\dfrac{A + B}{2}\right) \cos\left(\dfrac{A - B}{2}\right)$	$\sin A - \sin B = 2 \cos\left(\dfrac{A + B}{2}\right) \sin\left(\dfrac{A - B}{2}\right)$
$\cos A + \cos B = 2 \cos\left(\dfrac{A + B}{2}\right) \cos\left(\dfrac{A - B}{2}\right)$	$\cos A - \cos B = -2 \sin\left(\dfrac{A + B}{2}\right) \sin\left(\dfrac{A - B}{2}\right)$
$\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$	$\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1 - x^2}}$
$\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1 + x^2}$	$\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1 + x^2}$
$\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{\lvert x \rvert \sqrt{x^2 - 1}}$	$\dfrac{d}{dx}(\csc^{-1} x) = -\dfrac{1}{\lvert x \rvert \sqrt{x^2 - 1}}$

1. Let $f(x) = 4x^3 + 3x^2 - 6x + 12$.
   Let m be the minimum of f on the interval $[-2, 1]$.
   Let M be the maximum of f on the interval $[-2, 1]$.
   Find $m + M$.

   (a) 1
   (b) 4
   (c) 1/2
   (d) 17
   (e) 13
   (f) 21 $\longrightarrow$ **CORRECT**
   (g) 30
   (h) f does not have a maximum or minimum.

   Solution: $f'(x) = 12\,x^2 + 6\,x - 6$
   Critical Points: $x = \frac{1}{2}, -1$. Max of 17 at $x = -1$. Min of 4 at $x = -2$. $M + m = 21$.

2. Let $f(x) = x^2 - 5x + 3$ on the interval $[-2, 4]$. Let A be the average slope of f on the interval. The Mean Value Theorem guarantees a point c in the interval $(-2, 4)$ satisfying some conditions that you should know.
   Find c.

   (a) -2
   (b) -1
   (c) 0
   (d) 1 $\longrightarrow$ **CORRECT**
   (e) 2
   (f) 3
   (g) 4
   (h) It is not possible to find a c.

   Solution: The average slope on the interval is -3. Solving $f'(x) = -3$ gives $x = 1$.

3. Let $f(x) = xe^{ax}$. Find a so that $x = 5$ is a critical point of f.

   (a) $a = -5$
   (b) $a = -1$
   (c) $a = -\frac{1}{5}$ $\longrightarrow$ **CORRECT**

(d) $a = 0$
(e) $a = 1$
(f) $a = 5$
(g) $a = e^5$
(h) $a = 5e^5$
(i) It is not possible to find such an a.

Solution: $f'(x) = (ax+1)e^{ax}$, which gives a critical point at $x = -\frac{1}{a}$. Solving $-\frac{1}{a} = 5$ gives $a = -\frac{1}{5}$.

4. Let

$$f(x) = 2x^3 + 3ax^2 - a^2x + 4$$

Find a so that f has an inflection point at $x = 3$.

(a) $-6 \longrightarrow$ **CORRECT**
(b) -4
(c) -3
(d) -2
(e) 0
(f) 2
(g) 3
(h) 4
(i) 6
(j) It is not possible to find such an a.

Solution: $f''(x) = 6a + 12x$. Setting this equal to 0 gives $x = -\frac{1}{2}a$. If f has an inflection point at $x = 3$ then $a = -6$

5. Let $f(x) = |x^2 - 4|$. Find all critical points.

(a) 0
(b) -2
(c) 2
(d) $-2, 2$
(e) $-2, 0, 2 \longrightarrow$ **CORRECT**
(f) There are no critical points

Solution: $f'(x) = \frac{2\left(x^2-4\right)x}{|x^2-4|}$. This gives critical points when

$$x^2 - 4 = 0 \quad \text{and} \quad \frac{d}{dx}(x^2 - 4) = 0$$

Thus we get $-2, 2, 0$

6. Suppose you know that $f(1) = -2$, $f'(1) = 0$ and $f''(1) = -1$.
Choose the correct statement(s).
You may assume that all derivatives of f exist everywhere.

   (a) f has a local minimum at $x = 1$
   (b) f has a local maximum at $x = 1$ $\longrightarrow$ **CORRECT**
   (c) f has an absolute minimum at $x = 1$
   (d) f has an absolute maximum at $x = 1$
   (e) f has an inflection point at $x = 1$
   (f) f has neither a maximum or minimum or inflection point at $x = 1$
   (g) None of the above are correct
   (h) More than one of the above is correct

   Solution: This is the second derivative test, which says that f must have a local maximum at $x = 1$.

7. Let $f(x) = x^{1/3}(x+1)^{2/3}$. Find all critical points of f.

   Let N be the number of critical points.
   Let S be the sum of these critcal points

   Find $N + S$

   (a) $-\infty$
   (b) 0
   (c) $\frac{1}{3}$
   (d) 1
   (e) $\frac{4}{3}$
   (f) $\frac{5}{3}$ $\longrightarrow$ **CORRECT**
   (g) 2
   (h) ∞ (there are infinitely many critical points)
   (i) There are no critical points

Solution: $f'(x) = \frac{3x+1}{3(x+1)^{\frac{1}{3}}x^{\frac{2}{3}}}$ and therefore there are three critical points: -1, $-\frac{1}{3}$ and 0. Thus, $N + S = \frac{5}{3}$

8. A function, $f(x)$, has derivatives:

$$\begin{aligned} f'(x) =& (x+1)^2(x-1)^2 \\ f''(x) =& 4x(x-1)(x+1) \end{aligned}$$

Let:

$$\begin{aligned} M &= \text{Number of local maximum} \\ m &= \text{Number of local minimum} \\ I &= \text{Number of inflection points} \end{aligned}$$

Find $M + 2m + 3I$.

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4
(f) 6
(g) 9 $\longrightarrow$ **CORRECT**
(h) 15

Solution: Critical points: $-1, 1$.
Possible inflection points: $-1, 0, 1$

We now make a sign chart:

f''	$-$	$+$	$-$	$+$
f'	$+$	$+$	$+$	$+$

x: -1, 0, 1

Thus there are no max/mins, but 3 inflection points. Thus, $M + 2m + 3I = 9$.

Use the following for Questions 9 and 10.

Suppose you know the following data about a differentiable function $f(x)$.
Also suppose that suppose all derivatives of f exist.

x	$f(x)$	$f'(x)$	$f''(x)$
2	1	0	-1
5	0	-1	0
9	-4	0	1

9. Which of the following statements are true about the function f?

   I. f must have a local maximum at $x = 2$.

   II. f must have a local minimum at $x = 2$.

   III. f must have a local maximum at $x = 9$.

   IV. f must have a local minimum at $x = 9$.

   (a) I only
   (b) II only
   (c) III only
   (d) IV only
   (e) I and III only
   (f) I and IV only $\longrightarrow$ **CORRECT**
   (g) II and III only
   (h) II and IV only
   (i) None of them are true

   Solution: $x = 2$ and $x = 9$ are critical points.
   Since $f''(2) = -1$, we must have a local max at $x = 2$.
   Since $f''(9) = 1$, we must have a local min at $x = 9$.

10. Which of the following statements are true about the function f?

    I. f must have an inflection point at $x = 5$

    II. f could have an inflection point at $x = 5$

    III. f must have an inflection point somewhere in the interval $(2, 9)$

    (a) I only
    (b) II only
    (c) III only
    (d) I and II only

(e) I and III only

(f) II and III only $\longrightarrow$ **CORRECT**

(g) I, II and III

(h) None of them are true

Solution: $x = 5$ is a possible inflection point, but there is no guarantee that it is the only possible inflection point between 2 and 5. Thus, f could have an inflection point at $x = 5$, but there doesn't necessarily have to be one there. But, the sign of f'' does change between $x = 2$ and $x = 5$, t hus there must be an inflection point somewhere between $x = 2$ and $x = 5$.

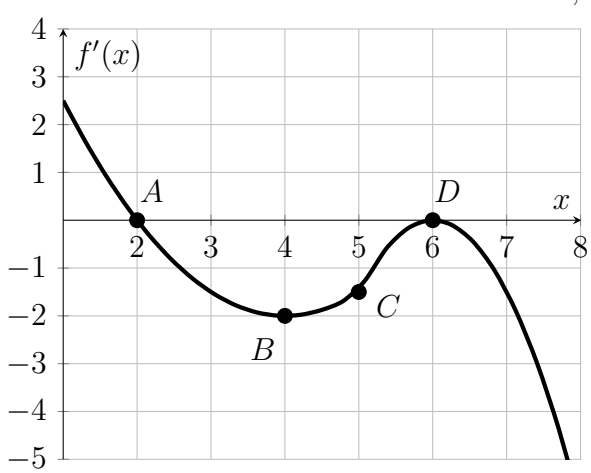
Here is a possible sign chart that would make $x = 5$ not an inflection point.

f''	$-$	$-$	$-$	$+$	$+$
f'	$+$	$-$	$-$	$-$	$+$

x: 2, 5, 7, 9

The point is that you have to test to see if it is an inflection point and there isn't enough data provided to check this.

The following will be used for Questions 11 and 12.
This is a graph of $f'(x)$.
Points $A = (2, 0)$, $B = (4, -2)$, $C = (5, -3/2)$ and $D = (6, 0)$ are labeled.
Determine the x-value of all local maxima, local minima and inflection points.



11. Where does $f(x)$ have a local maxima on the interval $[0, 8]$?

(a) $x = 2$ (Point A) $\longrightarrow$ **CORRECT**

(b) $x = 4$ (Point B)

(c) $x = 5$ (Point C)

(d) $x = 6$ (Point D)

(e) $f(x)$ has more than one local maxima

(f) $f(x)$ does not have any local maxima

(g) There is not enough information to determine the answer

Solution: Local maxima occur when f' changes from positive to negative. This happens at A.

12. Select which of the following are where $f(x)$ has an inflection point on the interval $[0, 8]$.

(a) $x = 2$ (Point A)

(b) $x = 4$ (Point B)

(c) $x = 5$ (Point C)

(d) $x = 6$ (Point D)

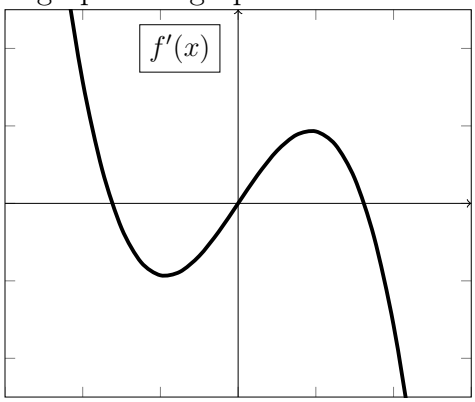
(e) $f(x)$ has more than one inflection point $\longrightarrow$ **CORRECT Points B and D**

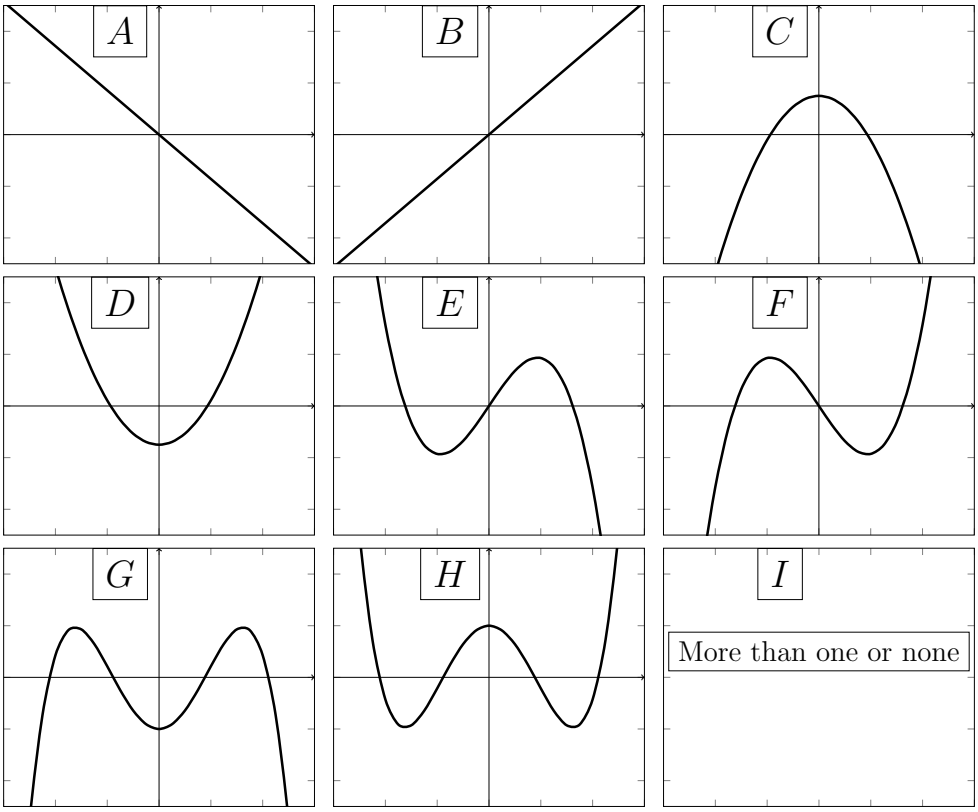
(f) $f(x)$ does not have any inflection points

(g) There is not enough information to determine the answer

Solution: Inflection points occur when the sign of f'' changes. We can read this off the graph of f' as the slope of f'. Thus, these slopes change from negative to positive at B and from positive to negative at D.

Use the following graph of $f'(x)$ and the answers given below for Questions 13 and 14.
All graphs are graphed in the same graphing window (same scales for all graphs).





13. Select the graph that represents $f(x)$.

    $\longrightarrow$ **CORRECT G**

    Solution: Where f'' crosses the x-axis, f must have a horizontal tangent line. This means only G and H and possible. Look at the signs of f' to make sure the graph is sloping how the graph of f' says f should, and you should see that G is the only possibility.

14. Select the graph that best represents $f''(x)$.

    $\longrightarrow$ **CORRECT** C

    Solution: Here we look for a graph that is zero where f' has horizontal tangent line. This is only C and D. Looking at signs, you can see that C is the only possibility.

15. Find

$$\lim_{x\to 1} \frac{\ln x}{\sin(\pi x)}$$

    (a) $-\pi$
    (b) -1
    (c) $-1/\pi$ $\longrightarrow$ **CORRECT**
    (d) 0

(e) 1

(f) $1/\pi$

(g) π

(h) DNE: Limit does not exist

Solution: Check to see that it is a 0/0 indeterminate form and use L'Hopital's Rule:

$$\lim_{x\to 1} \frac{\ln x}{\sin(\pi x)} \overset{LH}{=} \lim_{x\to 1} \frac{(1/x)}{\pi\cos(\pi x)} = \frac{1}{\pi(-1)} = -\frac{1}{\pi}$$

16. Find

$$\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^{x/5}$$

(a) 0

(b) $\frac{2}{5}$

(c) 1

(d) $e^{2/5}$ $\longrightarrow$ **CORRECT**

(e) $\frac{5}{2}$

(f) e

(g) e^2

(h) DNE: Limit does not exist

Solution: Check first that this is indeterminant, in the form 1^∞. Thus, we need to convert to a form that we can use L'Hopital with. Let $y = \left(1+\frac{2}{x}\right)^{x/5}$, and therefore $\ln y = (x/5)\ln\left(1+\frac{2}{x}\right)$. This is of the form $\infty\cdot 0$. So, we can do this:

$$\begin{aligned}\lim_{x\to\infty} \ln(y) &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{5/x} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{1+2/x}\cdot(-2/x^2)}{-5/x^2} \\ &= \lim_{x\to\infty} \frac{2}{5(1+2/x)} = \frac{2}{5}\end{aligned}$$

This gives us that, $\lim_{x\to\infty} \ln y = \frac{2}{5}$ and thus $\lim_{x\to\infty} y = e^{2/5}$.

17. Find

$$\lim_{x\to 1} \frac{x^2-4x+3}{x^2+3x-5}$$

(a) $-5/2$

(b) -1

(c) $-2/5$

(d) $0 \longrightarrow$ **CORRECT**

(e) $2/5$

(f) 1

(g) $5/2$

(h) DNE: Limit does not exist

Solution: This is not indeterminate, just plug in.

$$\lim_{x\to 1} \frac{x^2-4x+3}{x^2+3x-5} = \frac{0}{-1} = 0$$

18. Find

$$\lim_{x\to 0^+} x^{(x^2)}$$

(a) 0

(b) $\frac{1}{2}$

(c) $1 \longrightarrow$ **CORRECT**

(d) 2

(e) e

(f) e^2

(g) DNE: Limit does not exist

Solution: This is indeterminant of the form 0^0. Let $y = x^{(x^2)}$ and then $\ln y = x^2 \ln x$. Then $\lim_{x\to 0^+} \ln y$ is indeterminant of the form $0 \cdot \infty$.

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{LH}{=} \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} -\frac{x^2}{2} = 0$$

This gives us that, $\lim_{x\to\infty} \ln y = 0$ and thus $\lim_{x\to\infty} y = e^0 = 1$.

WRITTEN PROBLEM—SHOW YOUR WORK

Note: Your discussion section letter can be found on the front cover of exam.

Name:

ID: Discussion Section:

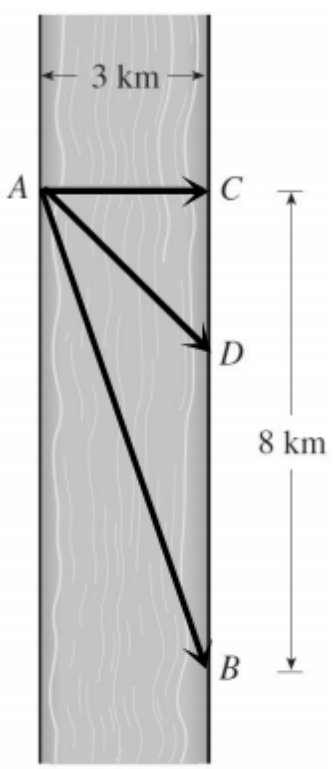
19. A man launches his boat from point A on a bank of a straight river (with no current), 3 km wide, and wants to reach point B, 8 km downstream on the oposite bank, as quickly as posible.
He could proced in one of three ways:

    I. Row his boat directly acros the river to point C and then run to B.

    II. Row directly to B.

    III. Row to some point D between C and B and then run to B.

    If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as posible? And, how much time will it take him to arrive at B?



Solution: Let x be the distance between C and D. Then,

Distance Rowed $=\sqrt{3^2+x^2}$

Distance Run $=8-x$

And thus we can set up time it takes

$$\begin{aligned} T =& \frac{\text{Distance Rowed}}{6} + \frac{\text{Distance Run}}{8} \\ =& \frac{\sqrt{9+x^2}}{6} + \frac{8-x}{8} \end{aligned}$$

x must be between 0 and 8, $x \in [0, 8]$. Take the derivative, find critical points, and test.

$$\frac{dT}{dt} = \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8}$$

Setting $\frac{dT}{dt} = 0$ gives critical point $x = \frac{9}{\sqrt{7}} \approx 3.402$.
Test critical points

x	y	
0	$3/2 = 1.5$	Max
$9/\sqrt{7}$	$1 + \sqrt{7}/8 \approx 1.3307$	Min
8	$\sqrt{73}/6 \approx 1.42$	

Thus, the fastest route is to aim for the point that is $1 + \sqrt{7}/8$ down river from the point directly across the river.

WRITTEN PROBLEM—SHOW YOUR WORK

Note: Your discussion section letter can be found on the front cover of exam.

Name:

ID: Discussion Section:

20. Let $f(x) = x^4 - 8x^3 + 18x^2 - 8$.

    (a) Find all critical points, maxima, minima, possible inflection points, and inflection points.

    (b) Draw the graph. Be sure to label all important points and graph features on your graph.

Solution: $f'(x) = 4\,(x-3)^2 x$
$f''(x) = 12\,(x-1)(x-3)$
Critical Points: $3, 0$
Possible inflection points: $3, 1$

f''	$+$	$+$	$-$	$+$
f'	$-$	$+$	$+$	$+$

x: 0, 1, 3

There are no asymptotes.

Get our y-values:

x	0	1	3
y	-8	3	19

The graph looks like.