Warm-up Problems - February 2, 2007
Solutions

1. Let \( f(x, y) = x^4 - 4xy + y^4 \)
   
   (a) \( f_x = 4x^3 - 4y \)
   
   (b) \( f_y = 4y^3 - 4x \)
   
   (c) \( f_{xx} = 12x^2 \)
   
   (d) \( f_{yy} = 12y^2 \)
   
   (e) \( f_{xy} = -4 \)
   
   (f) \( D = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16 = 16(9x^2y^2 - 1) \)

2. List the steps discussed in class for finding the maximum and minimum of a function \( f(x, y) \).
   Solution:
   
   I. Find the critical points by setting the first partials to zero:
   
   \( f_x(x, y) = 0 \quad f_y(x, y) = 0 \)
   
   II. Test the critical points:
   
   - Find \( D = f_{xx}f_{yy} - (f_{xy})^2 \)
   - Plug each critical point into \( D \).
     
     - If \( D > 0 \) then the critical point is either a maximum or a minimum.
       
       * If \( f_{xx} > 0 \) then the critical point is a minimum
       
       * If \( f_{xx} < 0 \) then the critical point is a maximum
     
     - If \( D < 0 \) then the critical point is a saddle.
     
     - If \( D = 0 \) then the test fails and you have to find another approach.

3. Find all critical points of the functions
   
   (a) \( f(x, y) = x^2 + xy + y^2 - 3x \)
       Solution: \( (2, -1) \)
   
   (b) \( f(x, y) = xy - x^3 - y^2 \)
       Solution: \( (0, 0) \) and \( (\frac{1}{6}, \frac{1}{12}) \)

4. Find if the critical points found in Problem 3 are maximums, minimums or saddles.
   
   (a) \( D(2, -1) = 3 \) and \( f_{xx}(2, -1) = 2 \) so we have a minimum at \( (2, -1) \).
   
   (b) \( D(0, 0) = -1 \) so we have a saddle at \( (0, 0) \).
       \( D\left(\frac{1}{6}, \frac{1}{12}\right) = 1 \) and \( f_{xx}\left(\frac{1}{6}, \frac{1}{12}\right) = -1 \) so we have a maximum at \( \left(\frac{1}{6}, \frac{1}{12}\right) \).
Lecture Problems

5. Find the area between \( y = x^2 - 3x \) and \( y = 2x \)

Solution: They intersect at \( x = 0, 5 \) and the area is \( \frac{125}{6} \)

6. Income stream has a flow rate of \( f(t) = 10e^{0.1t} \) dollars per year. Find the income produced by this income stream in the first two years.

Solution:

\[
\text{Income} = \int_0^2 10e^{0.1t} \, dt = 10e^{0.2} - 100 \approx 22.14
\]

7. Income stream has a flow rate of \( f(t) = 10e^{0.1t} \) dollars per year. The money is invested at \( 6\% \) interest compounded continuously. Find the future values of this income stream in the first two years.

Solution:

\[
\text{FV} = \int_0^2 10e^{0.1t}e^{0.04(2-t)} \, dt = e^{.12}(250e^{0.8} - 250) \approx 23.48
\]

8. Compute the integrals

(a)

\[
\int 2^x \, dx = \frac{e^{x\ln 2}}{\ln 2} = \frac{2^x}{\ln 2}
\]

(b)

\[
\int \frac{3x^5 - 2}{(x^6 - 4x + 12)^3} \, dx = \frac{1}{2} \int \frac{du}{u^{3/4}} = -\frac{1}{66(x^6 - 4x + 12)^{3/4}}
\]