Warm-up Problems - March 29, 2006
Solutions

1. Find the Taylor series for the functions below

   (a) $f(x) = \dfrac{1}{1+x^4}$ at $x = 0$

   Solution: You use the Taylor series for $\dfrac{1}{1-x}$ and plug in a $-x^4$ for x:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \\
\frac{1}{1+x^4} &= 1 + (-x^4) + (-x^4)^2 + (-x^4)^3 + (-x^4)^4 + \cdots \\
&= \sum_{n=0}^{\infty} (-x^4)^n \\
&= 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n x^{4n}
\end{aligned}$$

   (b) $f(x) = \dfrac{x}{1+x^4}$ at $x = 0$

   Solution: Use the previous problem and multiply through by x:

$$\begin{aligned}
\frac{1}{1+x^4} &= x - x^5 + x^9 - x^{13} + x^{17} - \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n x^{4n+1}
\end{aligned}$$

Lecture Problems

2. Using Problem 1a, approximate the integral below to within .00001

$$\int_0^{0.5} \frac{1}{1+x^4}\, dx$$

How many terms of the series do you need?

Solution:

$$\begin{aligned}
\int_0^{0.5} \frac{1}{1+x^4}\, dx &= \int_0^{0.5} \left(1 - x^4 + x^8 - x^{12} + x^{16} - \cdots\right)\, dx \\
&= \left[x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \frac{x^{17}}{17} - \cdots\right]_0^{0.5} \\
&= (0.5) - \frac{(0.5)^5}{5} + \frac{(0.5)^9}{9} - \frac{(0.5)^{13}}{13} + \frac{(0.5)^{17}}{17} - \cdots \\
&\approx (0.5) - \frac{(0.5)^5}{5} + \frac{(0.5)^9}{9} \approx 0.493967
\end{aligned}$$

Notice that $\frac{(0.5)^9}{9} = 0.00022$ and $\frac{(0.5)^{13}}{13} = 0.0000094$ so we need to use the first three terms of this series:

3. Using Problem 1b, approximate the integral below to within .00001

$$\int_0^{0.5} \frac{x}{1+x^4}\, dx$$

How many terms of the series do you need?

Solution:

$$\begin{aligned}
\int_0^{0.5} \frac{x}{1+x^4}\, dx &= \int_0^{0.5} \left(x - x^5 + x^9 - x^{13} + x^{17} - \cdots\right)\, dx \\
&= \left[\frac{x^2}{2} - \frac{x^6}{6} + \frac{x^{10}}{10} - \frac{x^{14}}{14} + \frac{x^{18}}{18} - \cdots\right]_0^{0.5} \\
&= \frac{(0.5)^2}{2} - \frac{(0.5)^6}{6} + \frac{(0.5)^{10}}{10} - \frac{(0.5)^{14}}{14} + \frac{(0.5)^{18}}{18} - \cdots \\
&\approx \frac{(0.5)^2}{2} - \frac{(0.5)^6}{6} + \frac{(0.5)^{10}}{10} \approx 0.1224935
\end{aligned}$$

Notice that $\dfrac{(0.5)^{10}}{10} = 0.000098$ and $\dfrac{(0.5)^{14}}{14} = 0.0000044$ so we need to use the first three terms of this series: