Warm-up Problems - March 24, 2006
Solutions

1. Find the Taylor polynomials of degree 4 for each of the functions at the point requested.

   (a) $f(x) = \frac{1}{5-x}$ at $x = 2$.
   Solution:

   $$p_4 = \frac{1}{3} + \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 + \frac{1}{81}(x-2)^3 + \frac{1}{243}(x-2)^4$$

   (b) $f(x) = \frac{1}{5-x}$ at $x = -2$.
   Solution:

   $$p_4 = \frac{1}{7} + \frac{1}{49}(x+2) + \frac{1}{343}(x+2)^2 + \frac{1}{2401}(x+2)^3 + \frac{1}{16807}(x+2)^4$$

   (c) $f(x) = \sqrt{1-x}$ at $x = 0$.
   Solution:

   $$p_4 = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4$$

   (d) $f(x) = \sqrt{1-x}$ at $x = -3$.
   Solution:

   $$p_4 = 2 - \frac{1}{4}(x+3) - \frac{1}{64}(x+3)^2 - \frac{1}{512}(x+3)^3 - \frac{5}{16384}(x+3)^4$$

Lecture Problems

2. Find the general form for the Taylor series for Problem 1. (You will probably need another sheet of paper!)

Solution:

(a) $f(x) = \frac{1}{5-x}$ at $x = 2$.

n	$f^{(n)}(x)$	$f^{(n)}(2)$	a_n
0	$(5-x)^{-1}$	$\frac{1}{3}$	$\frac{1}{3}$
1	$(5-x)^{-2}$	$\frac{1}{3^2}$	$\frac{1}{3^2}$
2	$2(5-x)^{-3}$	$\frac{2}{3^3}$	$\frac{2}{3^2 \cdot 2}$
3	$2 \cdot 3(5-x)^{-4}$	$\frac{2 \cdot 3}{3^4}$	$\frac{2 \cdot 3}{3^4 \cdot 3!} = \frac{1}{3^4}$
4	$2 \cdot 3 \cdot 4(5-x)^{-5}$	$\frac{2 \cdot 3 \cdot 4}{3^5}$	$\frac{2 \cdot 3 \cdot 4}{3^5 \cdot 4!} = \frac{1}{3^5}$
n	$n!(5-x)^{-(n+1)}$	$\frac{n!}{3^{n+1}}$	$\frac{1}{3^{n+1}}$

$$\begin{aligned}\frac{1}{5-x} &= \frac{1}{3} + \frac{(x-2)}{3^2} + \frac{(x-2)^2}{3^3} + \frac{(x-2)^3}{3^4} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(x-2)^n}{3^{n+1}}\end{aligned}$$

(b) $f(x) = \frac{1}{5-x}$ at $x = -2$.

n	$f^{(n)}(x)$	$f^{(n)}(-2)$	a_n
0	$(5-x)^{-1}$	$\frac{1}{7}$	$\frac{1}{7}$
1	$(5-x)^{-2}$	$\frac{1}{7^2}$	$\frac{1}{7^2}$
2	$2(5-x)^{-3}$	$\frac{2}{7^3}$	$\frac{2}{7^2 \cdot 2}$
3	$2 \cdot 3(5-x)^{-4}$	$\frac{2 \cdot 3}{7^4}$	$\frac{2 \cdot 3}{7^4 \cdot 3!} = \frac{1}{7^4}$
4	$2 \cdot 3 \cdot 4(5-x)^{-5}$	$\frac{2 \cdot 3 \cdot 4}{7^5}$	$\frac{2 \cdot 3 \cdot 4}{7^5 \cdot 4!} = \frac{1}{7^5}$
n	$n!(5-x)^{-(n+1)}$	$\frac{n!}{7^{n+1}}$	$\frac{1}{7^{n+1}}$

$$\begin{aligned}\frac{1}{5-x} &= \frac{1}{7} + \frac{(x+2)}{7^2} + \frac{(x+2)^2}{7^3} + \frac{(x+2)^3}{7^4} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(x+2)^n}{7^{n+1}}\end{aligned}$$

(c) $f(x) = \sqrt{1-x}$ at $x = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	a_n
0	$(1-x)^{1/2}$	1	1
1	$-\frac{1}{2}(1-x)^{-1/2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
2	$-\dfrac{1}{2}\cdot\dfrac{1}{2}(1-x)^{-3/2}$	$-\dfrac{1}{2^2}$	$-\dfrac{1}{2^2\cdot 2!}$
3	$-\dfrac{1}{2}\cdot\dfrac{1}{2}\cdot\dfrac{3}{2}(1-x)^{-5/2}$	$-\dfrac{3}{2^3}$	$-\dfrac{3}{2^3\cdot 3!}$
4	$-\dfrac{1}{2}\cdot\dfrac{1}{2}\cdot\dfrac{3}{2}\cdot\dfrac{5}{2}(1-x)^{-7/2}$	$-\dfrac{3\cdot 5}{2^4}$	$-\dfrac{3\cdot 5}{2^4\cdot 4!}$
5	$-\dfrac{1\cdot 3\cdot 5\cdot 7}{2^5}(1-x)^{-9/2}$	$-\dfrac{1\cdot 3\cdot 5\cdot 7}{2^5}$	$-\dfrac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}$
n	$-\dfrac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}(1-x)^{-(2n-1)/2}$	$-\dfrac{1\cdot 3\cdot 5\cdot(2n-3)}{2^n}$	$-\dfrac{1\cdot 3\cdot 5\cdot(2n-3)}{2^n\cdot n!}$

$$\begin{aligned}
\sqrt{1-x} =& 1 - \frac{1}{2}x - \frac{1}{2^2\cdot 2}x^2 - \frac{3}{2^3\cdot 3!}x^3 - \frac{3\cdot 5}{2^4\cdot 4!}x^4 + \cdots \\
=& 1 - \frac{1}{2}x - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot(2n-3)}{2^n\cdot n!}x^n \\
=& 1 - \frac{1}{2}x - \sum_{n=2}^{\infty} \frac{(2n-3)!}{(2^n\cdot n!)\left(2^{n-2}(n-2)!\right)}x^n \\
=& 1 - \frac{1}{2}x - \sum_{n=2}^{\infty} \frac{(2n-3)!}{2^{2n-2}n!(n-2)!}x^n
\end{aligned}$$

(d) $f(x) = \sqrt{1-x}$ at $x = -3$.

n	$f^{(n)}(x)$	$f^{(n)}(-3)$	a_n
0	$(1-x)^{1/2}$	1	1
1	$-\frac{1}{2}(1-x)^{-1/2}$	$-\frac{1}{2\cdot 2}$	$-\frac{1}{2\cdot 2}$
2	$-\frac{1}{2}\cdot\frac{1}{2}(1-x)^{-3/2}$	$-\frac{1}{2^2\cdot 2^3}$	$-\frac{1}{2^2\cdot 2^3\cdot 2!}$
3	$-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}(1-x)^{-5/2}$	$-\frac{3}{2^3\cdot 2^5}$	$-\frac{3}{2^3\cdot 2^5\cdot 3!}$
4	$-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}(1-x)^{-7/2}$	$-\frac{3\cdot 5}{2^4\cdot 2^7}$	$-\frac{3\cdot 5}{2^4\cdot 2^7\cdot 4!}$
5	$-\frac{1\cdot 3\cdot 5\cdot 7}{2^5}(1-x)^{-9/2}$	$-\frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 2^9}$	$-\frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 2^9\cdot 5!}$
n	$-\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}(1-x)^{-(2n-1)/2}$	$-\frac{1\cdot 3\cdot 5\cdot(2n-3)}{2^n\cdot 2^{2n-1}}$	$-\frac{1\cdot 3\cdot 5\cdot(2n-3)}{2^n\cdot 2^{2n-1}\cdot n!}$

$$\begin{aligned}
\sqrt{1-x} =& 2 - \frac{1}{4}(x+3) - \frac{1}{2^2\cdot 2^3\cdot 2}(x+3)^2 - \frac{3}{2^3\cdot 2^5\cdot 3!}(x+3)^3 - \frac{3\cdot 5}{2^4\cdot 2^7\cdot 4!}(x+3)^4 + \cdots \\
=& 2 - \frac{1}{4}(x+3) - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot(2n-3)}{2^n 2^{2n-1}\cdot n!}(x+3)^n \\
=& 2 - \frac{1}{4}(x+3) - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot(2n-3)}{2^{3n-1}\cdot n!}(x+3)^n \\
=& 2 - \frac{1}{4}(x+3) - \sum_{n=2}^{\infty} \frac{(2n-3)!}{2^{3n-1}\cdot n!\cdot 2^{n-2}(n-2)!}(x+3)^n \\
=& 2 - \frac{1}{4}(x+3) - \sum_{n=2}^{\infty} \frac{(2n-3)!}{2^{4n-3}n!(n-2)!}(x+3)^n
\end{aligned}$$

3. Find the Taylor series for the functions below.

(a)

$$\begin{aligned}
\frac{1}{1+x} =& \frac{1}{1-(-x)} \\
=& 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots \\
=& \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{1}{1+x^3} =& \frac{1}{1-(-x^3)} \\
=& 1 - x^3 + x^6 - x^9 + x^{12} - x^{15} + \cdots \\
=& \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{x^2}{1+x^3} =& x^2 \left(\frac{1}{1+x^3} \right) \\
=& x^2 \left(1 - x^3 + x^6 - x^9 + x^{12} - x^{15} + \cdots \right) \\
=& x^2 - x^5 + x^8 - x^{11} + x^{14} - x^{17} + \cdots \\
=& \sum_{n=0}^{\infty} (-1)^n x^{3n+2}
\end{aligned}$$

(d)

$$\begin{aligned}
x(e^x - 1) =& x \left(x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \right) \\
=& x^2 + \frac{1}{2}x^3 + \frac{1}{3!}x^4 + \frac{1}{4!}x^5 + \cdots \\
=& \sum_{n=1}^{\infty} \frac{1}{n!} x^{n+1}
\end{aligned}$$