Warm-up Problems - March 24, 2006
Solutions

1. Find the Taylor polynomials of degree 4 for each of the functions at the point requested.
   
   (a) \( f(x) = \frac{1}{5-x} \) at \( x = 2 \).
   Solution:
   \[ p_4 = \frac{1}{3} + \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 + \frac{1}{81}(x-2)^3 + \frac{1}{243}(x-2)^4 \]

   (b) \( f(x) = \frac{1}{5-x} \) at \( x = -2 \).
   Solution:
   \[ p_4 = \frac{1}{7} + \frac{1}{49}(x+2) + \frac{1}{343}(x+2)^2 + \frac{1}{2401}(x+2)^3 + \frac{1}{16807}(x+2)^4 \]

   (c) \( f(x) = \sqrt{1-x} \) at \( x = 0 \).
   Solution:
   \[ p_4 = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 \]

   (d) \( f(x) = \sqrt{1-x} \) at \( x = -3 \).
   Solution:
   \[ p_4 = 2 - \frac{1}{4}(x+3) - \frac{1}{64}(x+3)^2 - \frac{1}{512}(x+3)^3 - \frac{5}{16384}(x+3)^4 \]
Lecture Problems

2. Find the general form for the Taylor series for Problem 1. 
(You will probably need another sheet of paper!)

Solution:

(a) \( f(x) = \frac{1}{5-x} \) at \( x = 2 \).

\[
\begin{array}{|c|c|c|c|}
\hline
n & f^{(n)}(x) & f^{(n)}(2) & a_n \\
\hline
0 & (5 - x)^{-1} & \frac{1}{3} & \frac{1}{3} \\
1 & (5 - x)^{-2} & \frac{1}{3^2} & \frac{1}{3^2} \\
2 & 2(5 - x)^{-3} & \frac{2}{3^3} & \frac{2}{3^3} \\
3 & 2 \cdot 3(5 - x)^{-4} & \frac{2 \cdot 3}{3^4} & \frac{2 \cdot 3}{3^4} = \frac{1}{3^4} \\
4 & 2 \cdot 3 \cdot 4(5 - x)^{-5} & \frac{2 \cdot 3 \cdot 4}{3^5} & \frac{2 \cdot 3 \cdot 4}{3^5} = \frac{1}{3^5} \\
n & n!(5 - x)^{-(n+1)} & \frac{n!}{3^{n+1}} & \frac{1}{3^{n+1}} \\
\hline
\end{array}
\]

\[
\frac{1}{5-x} = \frac{1}{3} + \frac{(x-2)}{3^2} + \frac{(x-2)^2}{3^3} + \frac{(x-2)^3}{3^4} + \cdots \\
= \sum_{n=0}^{\infty} \frac{(x-2)^n}{3^{n+1}}
\]

(b) \( f(x) = \frac{1}{5-x} \) at \( x = -2 \).

\[
\begin{array}{|c|c|c|c|}
\hline
n & f^{(n)}(x) & f^{(n)}(-2) & a_n \\
\hline
0 & (5 - x)^{-1} & \frac{1}{7} & \frac{1}{7} \\
1 & (5 - x)^{-2} & \frac{1}{7^2} & \frac{1}{7^2} \\
2 & 2(5 - x)^{-3} & \frac{2}{7^3} & \frac{2}{7^3} \\
3 & 2 \cdot 3(5 - x)^{-4} & \frac{2 \cdot 3}{7^4} & \frac{2 \cdot 3}{7^4} = \frac{1}{7^4} \\
4 & 2 \cdot 3 \cdot 4(5 - x)^{-5} & \frac{2 \cdot 3 \cdot 4}{7^5} & \frac{2 \cdot 3 \cdot 4}{7^5} = \frac{1}{7^5} \\
n & n!(5 - x)^{-(n+1)} & \frac{n!}{7^{n+1}} & \frac{1}{7^{n+1}} \\
\hline
\end{array}
\]

\[
\frac{1}{5-x} = \frac{1}{7} + \frac{(x+2)}{7^2} + \frac{(x+2)^2}{7^3} + \frac{(x+2)^3}{7^4} + \cdots \\
= \sum_{n=0}^{\infty} \frac{(x+2)^n}{7^{n+1}}
\]
(c) $f(x) = \sqrt{1-x}$ at $x = 0.$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(0)$</th>
<th>$a_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$(1-x)^{1/2}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$-\frac{1}{2}(1-x)^{-1/2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>$-\frac{1}{2} \cdot \frac{1}{2}(1-x)^{-3/2}$</td>
<td>$-\frac{3}{4}$</td>
<td>$\frac{3}{4}$</td>
</tr>
<tr>
<td>3</td>
<td>$-\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2}(1-x)^{-5/2}$</td>
<td>$-\frac{24}{4}$</td>
<td>$\frac{24}{4}$</td>
</tr>
<tr>
<td>4</td>
<td>$-\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{2}(1-x)^{-7/2}$</td>
<td>$-\frac{24}{4}$</td>
<td>$\frac{24}{4}$</td>
</tr>
<tr>
<td>5</td>
<td>$-\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}(1-x)^{-9/2}$</td>
<td>$-\frac{24}{4}$</td>
<td>$\frac{24}{4}$</td>
</tr>
<tr>
<td>$n$</td>
<td>$-\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}(1-x)^{(2n-1)/2}$</td>
<td>$-\frac{1 \cdot 3 \cdot 5 \cdot (2n-3)}{2^n}$</td>
<td>$-\frac{1 \cdot 3 \cdot 5 \cdot (2n-3)}{2^n}$</td>
</tr>
</tbody>
</table>

$\sqrt{1-x} = 1 - \frac{1}{2} x - \frac{1}{2^2 \cdot 2} x^2 - \frac{3}{2^3 \cdot 3!} x^3 - \frac{3 \cdot 5}{2^4 \cdot 4!} x^4 + \cdots$

$= 1 - \frac{1}{2} x - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot (2n-3)}{2^n \cdot n!} x^n$

$= 1 - \frac{1}{2} x - \sum_{n=2}^{\infty} \frac{(2n-3)!}{(2^n \cdot n!) \cdot (2^{n-2}(n-2)!)} x^n$

$= 1 - \frac{1}{2} x - \sum_{n=2}^{\infty} \frac{(2n-3)!}{2^{2n-2}n!(n-2)!} x^n$
(d) \( f(x) = \sqrt{1-x} \) at \( x = -3. \)

\[
\begin{array}{|c|c|c|c|}
\hline
n & f^{(n)}(x) & f^{(n)}(-3) & a_n \\
\hline
0 & (1-x)^{1/2} & 1 & 1 \\
1 & -\frac{1}{2}(1-x)^{-1/2} & -\frac{1}{2} & -\frac{1}{2} \\
2 & -\frac{1}{2} \cdot \frac{1}{2} (1-x)^{-3/2} & -\frac{1}{2} \cdot \frac{3}{2} & -\frac{1}{2} \cdot \frac{3}{2} \cdot 2!
\hline
3 & -\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} (1-x)^{-5/2} & -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} & -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot 3!
\hline
4 & -\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} (1-x)^{-7/2} & -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} & -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot 4!
\hline
5 & -\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} (1-x)^{-9/2} & -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2} & -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2} \cdot 5!
\hline
n & \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} (1-x)^{-2n+2n-1/2} & -\frac{1 \cdot 3 \cdot 5 \cdot (2n-3)}{2^n \cdot 2^{2n-1}} & -\frac{1 \cdot 3 \cdot 5 \cdot (2n-3)}{2^n \cdot 2^{2n-1} \cdot n!}
\hline
\end{array}
\]

\[
\sqrt{1-x} = 2 - \frac{1}{4} (x+3) - \frac{1}{2^2 \cdot 2^3 \cdot 2} (x+3)^2 - \frac{3}{2^3 \cdot 2^5 \cdot 3!} (x+3)^3 - \frac{3 \cdot 5}{2^4 \cdot 2^7 \cdot 4!} (x+3)^4 + \cdots
\]

\[
= 2 - \frac{1}{4} (x+3) - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot (2n-3)}{2^n \cdot 2^{2n-1} \cdot n!} (x+3)^n
\]

\[
= 2 - \frac{1}{4} (x+3) - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot (2n-3)}{2^{3n-1} \cdot n!} (x+3)^n
\]

\[
= 2 - \frac{1}{4} (x+3) - \sum_{n=2}^{\infty} \frac{(2n-3)!}{2^{3n-1} n! \cdot 2^{n-2} (n-2)!} (x+3)^n
\]

\[
= 2 - \frac{1}{4} (x+3) - \sum_{n=2}^{\infty} \frac{(2n-3)!}{2^{3n-3} n! (n-2)!} (x+3)^n
\]
3. Find the Taylor series for the functions below.

(a) \[
\frac{1}{1 + x} = \frac{1}{1 - (-x)} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n
\]

(b) \[
\frac{1}{1 + x^3} = \frac{1}{1 - (-x^3)} = 1 - x^3 + x^6 - x^9 + x^{12} - x^{15} + \cdots = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}
\]

(c) \[
\frac{x^2}{1 + x^3} = x^2 \left( \frac{1}{1 + x^3} \right) = x^2 \left( 1 - x^3 + x^6 - x^9 + x^{12} - x^{15} + \cdots \right) = x^2 - x^5 + x^8 - x^{11} + x^{14} - x^{17} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{3n+2}
\]

(d) \[
x(e^x - 1) = x \left( x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots \right) = x^2 + \frac{1}{2} x^3 + \frac{1}{3!} x^4 + \frac{1}{4!} x^5 + \cdots = \sum_{n=1}^{\infty} \frac{1}{n!} x^{n+1}
\]