Warm-up Problems - March 10, 2006
Solutions

1. Chris plans to make an initial deposit into an account that pays 0.05\% interest, compounded continuously. He then plans to make withdrawls at a rate of $5000 per year.

(a) Let $y$ be the amount of money in the account.
Let $D$ be the amount of the initial deposit.
Set up the differential equations.

**Solution:**

$$y' = 0.05y - 5000, \quad y(0) = D \quad \text{(the initial deposit)}$$

(b) Solve the differential equation.

**Solution:**

$$I = e^{-t/20}$$

$$e^{-t/20}y' - \frac{1}{20}e^{-t/20}y = -5000$$

$$\left(e^{-t/20}y\right)' = -5000e^{-t/20}$$

$$e^{-t/20}y = 100,000e^{-t/20} + C$$

$$y = 100,000 + Ce^{t/20}$$

$$C = D - 100,000 \implies y = 100,000 + (D - 100,000)e^{t/20}$$

(c) If Chris wants to be able to be able to withdraw for 10 years, how much should the initial deposit be?

**Solution:** In this case, we want $t = 10$ to correspond to $y = 0$ (run out of money at 10 years). So, we solve for $D$:

$$0 = 100,000 + (D - 100,000)e^{10/20}$$

$$0 = 100,000 + (D - 100,000)e^{1/2}$$

$$D = - 100,000e^{-1/2} + 100,000 = 100,000 \left(1 - e^{-1/2}\right)$$

$$\approx$39,347
Lecture Problems

2. Mixture problem:

• 1000 gallon tank
• Water in: 10 gallons per hour. Contains 1.5 pounds of pollution per gallon.
• Water out: 8 gallons per hour. Flows out completely mixed.
• Initially the tank contains 500 gallons, 4 pounds of pollution per gallon.

Let $p$ be the amount (in pounds) of pollution in the tank at time $t$ hours.

3. Find $p(0)$.

**Solution:** This is the amount of pollution in the tank initially:

$$p(0) = \left( 4 \text{ lbs/gallon} \right) (500 \text{ gallons}) = 2000 \text{ lbs}$$

4. Find the volume of the tank at any time.
(If you are having difficulty with this question, determine the volume at $t = 0, t = 1, t = 2, t = 3, \text{ etc.}$)

**Solution:**

$$V(t) = 500 + 2t$$

5. What is the rate (in pounds per hour) that pollution is entering the tank?

**Solution:**

$$\text{rate in} = \left( 10 \text{ gallons/hour} \right) \left( 1.5 \text{ lbs/gallon} \right) = 15 \text{ lbs/hour}$$

6. At any given time $t$, what is the concentration of the pollution in the tank (pounds per gallon)?

**Solution:**

$$\text{concentration} = \frac{\text{pounds of pollution}}{\text{amt liquid}} = \frac{p}{V} = \frac{p}{500 + 2t}$$

7. What is the rate (in pounds per hour) that pollution is leaving the tank?

**Solution:**

$$\text{rate out} = \left( \text{concentration} \right) \left( \text{rate of water out} \right) = \left( \frac{p}{500 + 2t} \right) 8 = \frac{8p}{500 + 2t}$$

8. What is the differential equation for $p' = \frac{dp}{dt}$?

**Solution:**

$$p' = (\text{rate in}) - (\text{rate out}) = 15 - \frac{8p}{500 + 2t} = 15 - \frac{4p}{250 + t}$$
9. What is the general solution to the differential equation?

Solution: Integrating factor: \( I = (250 + t)^4 \)

\[
p' + \frac{4p}{250 + t} = 15
\]

\[
(250 + t)^4 p' + 4(250 + t)^3 p = 15(250 + t)^4
\]

\[
[(250 + t)^4 p]' = 15(250 + t)^4
\]

\[
(250 + t)^4 p = 3(250 + t)^5 + C
\]

\[
p = 3(250 + t) + \frac{C}{(250 + t)^4}
\]

10. What is the initial condition and what is the particular solution to the differential equation?

Solution: The initial condition is \( p(0) = 2000 \) which gives \( C = 1250(250)^4 = 2^{55}16 \)

\[
p = 3(250 + t) + \frac{2^{55}16}{(250 + t)^4}
\]

11. How much pollution is in the tank after 10 hours? (in pounds)

Solution: Plug in \( t = 10 \):

\[
p(10) = 3(250 + 10) + \frac{2^{55}16}{(250 + 10)^4} \approx 1848.5
\]

12. What is the concentration of the pollution after 10 hours? (in pounds per gallon)

Solution: Plug in \( t = 10 \) and divide by the volume:

\[
\text{Concentration} = \frac{p(10)}{V(10)} \approx \frac{1848.5}{520} \approx 3.55 \text{ lbs/gallon}
\]

13. What is the concentration in the tank when the tank starts spilling over?

Solution: This occurs when \( V = 1000 \), or when \( t = 250 \). We have \( p(250) \approx 1578 \) and the concentration is therefore approximately 1.58 pounds per gallon.