April 28, 2006 — Final Exam Review Problems Solutions

1. Find the Taylor series for $f(x) = x^3 - e^{2x}$ at $x = 0$

$$\begin{aligned} f(x) =& x^3 - \left[1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \frac{(2x)^5}{5!} + \cdots\right] \\ =& -1 - 2x - \frac{2^2x^2}{2!} - \left(\frac{2^3}{3!} - 1\right)x^3 - \frac{2^4x^4}{4!} - \frac{2^5x^5}{5!} - \cdots \end{aligned}$$

2. Find all Taylor polynomials of degree less than and equal to 5 for $f(x) = x^7 - 4x^6 + 3x^4 + 2x^3 - x^2 + 5x - 31$ at $x = 0$

Solution:

$$\begin{aligned} p_0 =& -31 \\ p_1 =& -31 + 5x \\ p_2 =& -31 + 5x - x^2 \\ p_3 =& -31 + 5x - x^2 + 2x^3 \\ p_4 =& -31 + 5x - x^2 + 2x^3 + 3x^4 \\ p_5 =& -31 + 5x - x^2 + 2x^3 + 3x^4 \end{aligned}$$

3. Find all Taylor polynomials of degree less than and equal to 5 for $f(x) = x^7 - 4x^6 + 3x^4 + 2x^3 - x^2 + 5x - 31$ at $x = 1$

Solution:

$$\begin{aligned} p_0 =& -25 \\ p_1 =& -25 + 4(x-1) \\ p_2 =& -25 + 4(x-1) - 16(x-1)^2 \\ p_3 =& -25 + 4(x-1) - 16(x-1)^2 - 31(x-1)^3 \\ p_4 =& -25 + 4(x-1) - 16(x-1)^2 - 31(x-1)^3 - 22(x-1)^4 \\ p_5 =& -25 + 4(x-1) - 16(x-1)^2 - 31(x-1)^3 - 22(x-1)^4 - 3(x-1)^5 \end{aligned}$$

4. Find all Taylor polynomials of degree less than and equal to 5 for $f(x) = x^7 - 4x^6 + 3x^4 + 2x^3 - x^2 + 5x - 31$ at $x = -3$

Solution:

$$\begin{aligned} p_0 =& -4969 \\ p_1 =& -4969 + 10676(x+3) \\ p_2 =& -4969 + 10676(x+3) - 9820(x+3)^2 \\ p_3 =& -4969 + 10676(x+3) - 9820(x+3)^2 + 4961(x+3)^3 \\ p_4 =& -4969 + 10676(x+3) - 9820(x+3)^2 + 4961(x+3)^3 - 1482(x+3)^4 \\ p_5 =& -4969 + 10676(x+3) - 9820(x+3)^2 + 4961(x+3)^3 - 1482(x+3)^4 + 261(x+3)^5 \end{aligned}$$

5. Use a degree 4 polynomial to approximate the integrals below. Estimate the error of your approximation.

$$\int_0^{1/2} e^{-x^2}\,dx \approx \frac{443}{960} \approx 0.461458$$

$|\text{Error}| \leq 0.000186$

6. Use a degree 4 polynomial to approximate the integrals below. Estimate the error of your approximation.

$$\int_0^{1/2} \ln(1+x^2)\,dx \approx \frac{37}{960} \approx 0.038542$$

$|\text{Error}| \leq 0.000372$

7. Approximate the integral below to within 0.0001. What is the degree of the Taylor polynomial that you used.

$$\int_0^{1/3} \frac{1}{1+x^2}\,dx \approx \frac{24628}{76545} \approx 0.3217$$

Use the taylor polynomial of degree 6

8. Approximate the integral below to within 0.0001. What is the degree of the Taylor polynomial that you used.

$$\int_0^{1/3} \frac{1}{1+x^4}\,dx \approx \frac{404}{1215} \approx 0.33251$$

Use the taylor polynomial of degree 4

9. An income stream of $\$1,000e^{-0.05t}$ per year for the next 5 years is invested as it arrives at a compound interest rate of 4%. What is the total income, future value and present value of this income stream?

   Solution:

$$
\begin{aligned}
TI &= \int_0^T f(t)\,dt \approx 4424 \\
FV &= e^{rT}\int_0^T f(t)e^{-rt}\,dt \approx 4918 \\
PV &= \int_0^T f(t)e^{-rt}\,dt \approx 4026
\end{aligned}
$$

10. An income stream of $\$1,000e^{-0.5t}$ per year for the next 5 years is invested as it arrives at a compound interest rate of 5%. What is the total income, future value and present value of this income stream?

    Solution:

$$
\begin{aligned}
TI &= \int_0^T f(t)\,dt \approx 4424 \\
FV &= e^{rT}\int_0^T f(t)e^{-rt}\,dt \approx 5052 \\
PV &= \int_0^T f(t)e^{-rt}\,dt \approx 3935
\end{aligned}
$$

11. Oil is produced from a well at a rate of

$$
R(t) = \frac{100}{t+10} + 10
$$

    in millions of barrels of oil per year. Find the total amount of oil produced in years $t = 1$ to $t = 4$.

    Solution:

$$
\text{Amount} = \int_1^4 R(t)\,dt \approx 54.12
$$

12. Oil is produced from a well at a rate of

$$
R(t) = \frac{100t}{t^2+10} + 10
$$

    in millions of barrels of oil per year. Find the total amount of oil produced in years $t = 1$ to $t = 4$.

    Solution:

$$
\text{Amount} = \int_1^4 R(t)\,dt \approx 73.01
$$

13. For the function, find all critical points. Use the second derivative test to test critical points. Find all maximum and minimum.

$$f(x,y) = 2x^2 - 2x^2y + 6y^3;$$

Solution:

Crit pts	f	D	f_{xx}	conclusion
$(0,0)$	0	-36	-2	saddle
$(9,-3)$	27	36	-2	max

14. Find the integral

$$\iint_R xe^{x+xy}\,dA$$

where R is the rectangle $R = \{(x,y) : 0 \le x \le 2, 0 \le y \le 1\}$

Solution:

$$-e^2 + \frac{e^4}{2} + \frac{1}{2} \approx 20.41$$

15. A new product was introduced one year ago and current sales are \$2 million per year. The sales grow at a rate proportional to the difference between sales and the projected maximum of \$8 million.

Find the differential equation and find the sale one year from now.

Solution: We need to find $y(2)$.

$$\frac{dy}{dt} = k(8-y) \qquad y(0) = 0, y(1) = 1$$

$$y = 8 + Ce^{-kt} = 8 - 8e^{t\ln(4/3)} = 8 - 8\left(\frac{4}{3}\right)^t$$

$$y(2) = 8 - 8\left(\frac{4}{3}\right)^2 = 3.5$$

16. Find the solution to the initial value problem:

$$2xy' + y = 4x^{3/4}$$
$$y(1) = 2$$

Solution:

$$y = \frac{8}{5}x^{3/4} + \frac{2}{5}x^{-1/2}$$

17. Find the solution to the initial value problem:

$$2xy' + y = 4x^{3/4}$$
$$y(1) = 4$$

Solution:

$$y = \frac{8}{5}x^{3/4} + \frac{12}{5}x^{-1/2}$$

18. Find the average value of the function $f(x, y) = x + \frac{1}{xy}$ on the rectangle $R = \{(x, y) : 2 \leq x \leq 5, 1 \leq y \leq 3\}$.

Solution:

$$\text{Ave} = \frac{1}{6}\iint_R x + \frac{1}{xy}\, dA \approx 3.6678$$