April 28, 2006 — Final Exam Review Problems

1. Find the Taylor series for \( f(x) = x^3 - e^{2x} \) at \( x = 0 \)

\[
f(x) = x^3 - \left[ 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \frac{(2x)^5}{5!} + \cdots \right]
\]

\[
= -1 - 2x - \frac{2^2 x^2}{2!} - \left( \frac{2^3}{3!} - 1 \right) x^3 - \frac{2^4 x^4}{4!} - \frac{2^5 x^5}{5!} - \cdots
\]

2. Find all Taylor polynomials of degree less than and equal to 5 for \( f(x) = x^7 - 4x^6 + 3x^4 + 2x^3 - x^2 + 5x - 31 \) at \( x = 0 \)

Solution:

\[
p_0 = -31 \\
p_1 = -31 + 5x \\
p_2 = -31 + 5x - x^2 \\
p_3 = -31 + 5x - x^2 + 2x^3 \\
p_4 = -31 + 5x - x^2 + 2x^3 + 3x^4 \\
p_5 = -31 + 5x - x^2 + 2x^3 + 3x^4
\]

3. Find all Taylor polynomials of degree less than and equal to 5 for \( f(x) = x^7 - 4x^6 + 3x^4 + 2x^3 - x^2 + 5x - 31 \) at \( x = 1 \)

Solution:

\[
p_0 = -25 \\
p_1 = -25 + 4(x - 1) \\
p_2 = -25 + 4(x - 1) - 16(x - 1)^2 \\
p_3 = -25 + 4(x - 1) - 16(x - 1)^2 - 31(x - 1)^3 \\
p_4 = -25 + 4(x - 1) - 16(x - 1)^2 - 31(x - 1)^3 - 22(x - 1)^4 \\
p_5 = -25 + 4(x - 1) - 16(x - 1)^2 - 31(x - 1)^3 - 22(x - 1)^4 - 3(x - 1)^5
\]

4. Find all Taylor polynomials of degree less than and equal to 5 for \( f(x) = x^7 - 4x^6 + 3x^4 + 2x^3 - x^2 + 5x - 31 \) at \( x = -3 \)

Solution:

\[
p_0 = -4969 \\
p_1 = -4969 + 10676(x + 3) \\
p_2 = -4969 + 10676(x + 3) - 9820(x + 3)^2 \\
p_3 = -4969 + 10676(x + 3) - 9820(x + 3)^2 + 4961(x + 3)^3 \\
p_4 = -4969 + 10676(x + 3) - 9820(x + 3)^2 + 4961(x + 3)^3 - 1482(x + 3)^4 \\
p_5 = -4969 + 10676(x + 3) - 9820(x + 3)^2 + 4961(x + 3)^3 - 1482(x + 3)^4 + 261(x + 3)^5
\]
5. Use a degree 4 polynomial to approximate the integrals below. Estimate the error of your approximation.

\[
\int_{0}^{\frac{1}{\sqrt{2}}} e^{-x^2} \, dx \approx \frac{443}{960} \approx 0.461458
\]

\[|\text{Error}| \leq 0.000186\]

6. Use a degree 4 polynomial to approximate the integrals below. Estimate the error of your approximation.

\[
\int_{0}^{\frac{1}{\sqrt{2}}} \ln(1 + x^2) \, dx \approx \frac{37}{960} \approx 0.038542
\]

\[|\text{Error}| \leq 0.000372\]

7. Approximate the integral below to within 0.0001. What is the degree of the Taylor polynomial that you used.

\[
\int_{0}^{\frac{1}{\sqrt{3}}} \frac{1}{1 + x^2} \, dx \approx \frac{24628}{76545} \approx 0.3217
\]

Use the taylor polynomial of degree 6

8. Approximate the integral below to within 0.0001. What is the degree of the Taylor polynomial that you used.

\[
\int_{0}^{\frac{1}{\sqrt{3}}} \frac{1}{1 + x^4} \, dx \approx \frac{404}{1215} \approx 0.33251
\]

Use the taylor polynomial of degree 4
9. An income stream of $1,000e^{-0.05t}$ per year for the next 5 years is invested as it arrives at a compound interest rate of 4%. What is the total income, future value and present value of this income stream?

Solution:

\[ TI = \int_0^T f(t) \, dt \approx 4424 \]

\[ FV = e^{rT} \int_0^T f(t)e^{-rt} \, dt \approx 4918 \]

\[ PV = \int_0^T f(t)e^{-rt} \, dt \approx 4026 \]

10. An income stream of $1,000e^{-0.5t}$ per year for the next 5 years is invested as it arrives at a compound interest rate of 5%. What is the total income, future value and present value of this income stream?

Solution:

\[ TI = \int_0^T f(t) \, dt \approx 4424 \]

\[ FV = e^{rT} \int_0^T f(t)e^{-rt} \, dt \approx 5052 \]

\[ PV = \int_0^T f(t)e^{-rt} \, dt \approx 3935 \]

11. Oil is produced from a well at a rate of

\[ R(t) = \frac{100}{t+10} + 10 \]

in millions of barrels of oil per year. Find the total amount of oil produced in years \( t = 1 \) to \( t = 4 \).

Solution:

\[ \text{Amount} = \int_1^4 R(t) \, dt \approx 54.12 \]

12. Oil is produced from a well at a rate of

\[ R(t) = \frac{100t}{t^2 + 10} + 10 \]

in millions of barrels of oil per year. Find the total amount of oil produced in years \( t = 1 \) to \( t = 4 \).

Solution:

\[ \text{Amount} = \int_1^4 R(t) \, dt \approx 73.01 \]
13. For the function, find all critical points. Use the second derivative test to test critical points. Find all maximum and minimum.

\[ f(x, y) = 2x^2 - 2x^2y + 6y^3; \]

**Solution:**

<table>
<thead>
<tr>
<th>Crit pts</th>
<th>( f )</th>
<th>( D )</th>
<th>( f_{xx} )</th>
<th>conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>0</td>
<td>-36</td>
<td>-2</td>
<td>saddle</td>
</tr>
<tr>
<td>(9, -3)</td>
<td>27</td>
<td>36</td>
<td>-2</td>
<td>max</td>
</tr>
</tbody>
</table>

14. Find the integral

\[ \int \int_R x e^{x^2+y} \, dA \]

where \( R \) is the rectangle \( R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 1\} \)

**Solution:**

\[ -e^2 + \frac{e^4}{2} + \frac{1}{2} \approx 20.41 \]

15. A new product was introduced one year ago and current sales are $2 million per year. The sales grow at a rate proportional to the difference between sales and the projected maximum of $8 million.

Find the differential equation and find the sale one year from now.

**Solution:** We need to find \( y(2) \).

\[ \frac{dy}{dt} = k(8 - y) \quad y(0) = 0, y(1) = 1 \]

\[ y = 8 + Ce^{-kt} = 8 - 8e^{t \ln(4/3)} = 8 - 8 \left(\frac{4}{3}\right)^t \]

\[ y(2) = 8 - 8 \left(\frac{4}{3}\right)^2 = 3.5 \]

16. Find the solution to the initial value problem:

\[ 2xy' + y = 4x^{3/4} \]

\[ y(1) = 2 \]

**Solution:**

\[ y = \frac{8}{5}x^{3/4} + \frac{2}{5}x^{-1/2} \]
17. Find the solution to the initial value problem:

\[ 2xy' + y = 4x^{3/4} \]
\[ y(1) = 4 \]

**Solution:**

\[ y = \frac{8}{5}x^{3/4} + \frac{12}{5}x^{-1/2} \]

18. Find the average value of the function \( f(x, y) = x + \frac{1}{xy} \) on the rectangle \( R = \{(x, y) : 2 \leq x \leq 5, 1 \leq y \leq 3\} \).

**Solution:**

\[ \text{Ave} = \frac{1}{6} \int \int_R x + \frac{1}{xy} \, dA \approx 3.6678 \]