April 21, 2006 — Final Exam Review Sheet

Topics for the Final Exam

I. Chapter 6-7: Integration
   (a) Integration by substitution
   (b) Differential equations — growth and decay (see Chapter 9)
   (c) Area between curves: Lorenz curves, Gini index of income concentration
   (d) Applications: continuous income streams (Total income, Future Value, Present Value)
   (e) Integration by parts ($\int u \, dv = uv - \int v \, du$)

II. Chapter 8: Multivariable Calculus
   (a) Partial derivatives
   (b) Maxima and minima, critical points, second derivative test
   (c) Lagrange multipliers
   (d) Least Squares (linear only)
   (e) Double integrals

III. Chapter 9: Differential equations
   (a) (No slope fields)
   (b) Separation of variables
   (c) First order linear
   (d) Finding the differential equation from the problem:
      i. Exponential growth, limited growth (no logistic growth)
      ii. Compound interest (with withdrawls/deposits)
      iii. Mixture problems

IV. Chapter 10: Taylor polynomials

(a) Taylor polynomials (find a Taylor polynomial of a given order)

(b) Using a Taylor polynomial to approximate an integral

   i. Use a Taylor polynomial of a given degree

   ii. Determine the Taylor polynomial of the correct degree to obtain an approximation of the desired accuracy.

   iii. Will not need to use the "Remained Theorem" but will need to know the alternating series estimate as well as when the alternating series estimate applies.

(c) Taylor Series: nothing on finding the general term of a series

V. Chapter 11: Probability and Statistics

(a) Probability density functions (is a function a pdf?)

(b) Cumulative distribution functions (pdf $\longrightarrow$ cdf, cdf $\longrightarrow$ pdf)

(c) Using a pdf/cdf to compute probabilities

(d) Expected value, variance, standard deviation, median: computing from a pdf.

(e) Exponential pdf: using.

(f) Normal pdf: using.

Review Problems

1. An income stream of \$1,000 per year for the next 5 years is invested as it arrives at a compound interest rate of 5%. What is the total income, future value and present value of this income stream?

   Solution:

$$\begin{aligned} TI &= \int_0^T f(t)\,dt = 5000 \\ FV &= e^{rT}\int_0^T f(t)e^{-rt}\,dt \approx 5681 \\ PV &= \int_0^T f(t)e^{-rt}\,dt \approx 4424 \end{aligned}$$

2. Find the Gini index of the following Lorenz Curves:

   (a) $f(x) = x^{3.8}$

   Solution: Index ≈ 0.583

   (b) $f(x) = \frac{3}{10}x + \frac{7}{10}x^3$

   Solution: Index $= 0.35$

   (c) $f(x) = x + 1$

   Solution: Trick question! This isn't a Lorenz curve!

   (d) $f(x) = xe^{x-1}$

   Solution: You will have to integrate by parts! Index $= 1 - 2e{-1} \approx 0.26$

   (e) $f(x) = x^2e^{x-1}$

   Solution: You will have to integrate by parts! Index $= 4e{-1} - 1 \approx 0.4715$

   (f) $f(x) = xe^{x^2-1}$

   Solution: Index $= e{-1} \approx 0.3679$

3. After a person take a pill, the drug contained in the pill is assimilated into the bloodstream. The rate of assimilation t minutes after taking the pill is

$$R(t) = te^{-0.2t}$$

Rind the total amount of drug assimilated into the bloodstream during the first 10 minutes after the pill is taken.

(Hint: integration by parts!)

Solution:

$$\text{Amount} = \int_0^{10} R(t)\, dt \approx 14.85$$

4. For the function, find all critical points. Use the second derivative test to test critical points. Find all maximum and minimum.

$$f(x, y) = 2x^2 + 4xy - 12x - 4y^2 - 20y + 31$$

Solution: Only critical point is at $(2, 1)$, which is a saddle.

5. Find the integral

$$\iint_R \sqrt{x+y}\, dA$$

where R is the rectangle $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2\}$

Solution: $\frac{4}{15}(3^{5/2} - 2^{5/2} - 1)$

6. A new product was introduced one year ago and current sales are \$1 million per year. The sales grow at a rate proportional to the difference between sales and the projected maximum of \$10 million.

   Find the differential equation and find the sale one year from now.

   Solution: We need to find $y(2)$.

$$\frac{dy}{dt} = k(10 - y) \qquad y(0) = 0, y(1) = 1$$
$$y = 10 - 10e^{t \ln 0.9}$$
$$y(2) = 10 - 10(0.9)^2 = 1.9$$

7. Find the solution to the initial value problem:

$$2xy' + y = 4x^{3/2}$$
$$y(1) = 2$$

   Find $y(4)$.

   Solution:

$$y = x^{3/2} + x^{-1/2}$$
$$y(4) = \frac{17}{2}$$

8. Find the average value of the function $f(x, y) = ye^{xy}$ on the rectangle $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2\}$.

   Solution:

$$\text{Ave} = \frac{1}{2} \iint_R ye^{xy}\, dA = \frac{1}{2}e^2 - \frac{3}{2}$$

9. The Time Travel Index is a measure of rush hour congestion. It is the ratio of rush hour drive time to free flow drive time for selected routes. The index for St. Louis for several recent years was:

Year	1982	1993	2002	2003
Index	1.09	1.18	1.24	1.22

Use linear regression to predict the value for 2007.

Solution: Ignore the stuff about the ratio, just compute a least squares line.

$$y = 1.2553$$

10. The Cobb-Douglas production function for a new product is given by

$$f(x, y) = 24x^{3/4}y^{1/4}$$

where x is the number of units of labor and y is the number of units of capital. Each unit of labor costs \$100 and each unit of capital costs \$50. If \$500,000 is budgeted for production, how many units of labor and how many units of capital should be purchased (to maximize the function).

(Lagrange multipliers) **Solution:** Use Lagrange multipliers to find $x = 3750$ and $y = 2500$.

11. A random variable X has pdf

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2 - 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Find the mean, median and standard deviation.

Solution:

$$\begin{aligned} m =& 1 - \frac{\sqrt{2}}{2} \\ \mu =& \frac{1}{3} \\ V(X) =& \frac{1}{18} \\ \sigma =& \frac{1}{3\sqrt{2}} \end{aligned}$$

12. A random variable has an exponential distribution with mean 3. What is the probability that $3 < X < 9$?

$$P(3 < X < 9) = F(9) - F(3) = e^{-1} - e^{-3} \approx 0.3181$$