# Primes

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Much of this is due to Joshua Zucker.

1. What are the integers made up of?
   (a) If your only operation is plus, "+", what integers do you need to make all the integers?
   (b) If your only operation is multiplication, what integers do you need to make all the positive integers?

2. What if we restricted ourselves to the even integers? In other words, 6 would be in our set but 3 would not. This work make 6 a "prime" since it can not be written as a product of two even integers.
   (a) What are the "primes" if our number system is the even integers?
   (b) In the set of even integers, what is a prime factorization of the following integers? In particular, see if you can find different prime factorizations.

$$4 =$$
$$8 =$$
$$12 =$$
$$16 =$$
$$20 =$$
$$24 =$$
$$28 =$$
$$32 =$$
$$36 =$$
$$40 =$$
$$60 =$$
$$420 =$$

   (c) What if we restricted to odd numbers? What would the "primes" be in this case?
   (d) (Back to the integers.) What does it mean for a number to have a unique prime factorization?

3. How many primes are there? There are lots of proofs that there are an inifinite number of primes many of which can be found here: `http://primes.utm.edu/notes/proofs/infinite/`.

   Here is one of the more popular proofs that there are an infinite number of primes (due to Euclid). Suppose there were only a finite number of primes. Then we could list them:

$$2, 3, 5, 7, 11, 13, \ldots, P$$

   where $P$ is the largest prime. Then, consider the number:

$$N = 2 \cdot 3 \cdot 5 \cdot 7 \cdots P + 1$$

   Then you notice that $N > P$ and if you divide $N$ by one of our primes we get a remainder of 1. In other words, none of our primes divides $N$ and therefore $N$ should have been on our list of primes

   (a) Consider the numbers (motivated by Euclid's proof) and determine if they are prime or composite (find the prime factorizations):

$$\begin{aligned}
&2 + 1 = 3 \\
&2 \cdot 3 + 1 = 7 \\
&2 \cdot 3 \cdot 5 + 1 = 31 \\
&2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211 \\
&2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 = 2311 \\
&2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 \\
&2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 + 1 = 510511 \\
&2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 + 1 = 9699691 \\
&2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 + 1 = 223092871 \\
&2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 + 1 = 6469693231
\end{aligned}$$

   (b) Would Euclid's proof have worked if instead of $2 \cdot 3 \cdots P + 1$ we had used

$$N = 2 \cdot 3 \cdot 5 \cdot 7 \cdots P - 1$$

   or perhaps even $P! + 1$? As in the previous question, try factoring this sequence of numbers.

4. Is 221 prime? If not what are its factors?

5. What is the largest divisor you need to check to be sure that 397 is prime? How do you know it is the largest?

6. Is 8171 prime? How do you know for certain?

7. List all the primes between 1 and 500 using the *Sieve of Eratosthenes*.

# Sieve of Eratosthenes

A *sieve* is like a strainer. We're going to strain out the primes.

To use this sieve, start with a long list of numbers (like the list below). Forget about 1, it is neither prime nor composite. You know that 2 is a prime, so circle it. Then, cross off all multiples of 2 $(4, 6, 8, 10, \ldots)$ since you know they cannot be prime. Then, circle the next number on the list that isn't crossed out (3) and circle it. Cross off all the multiple of that number and now repeat. (The next number to circle will be 5.) When you are done all the circled numbers will be prime.

Watch for patterns when you are doing all of this. It can be helpful to use different colored pencils.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 |
| 121 | 122 | 123 | 124 | 125 | 126 | 127 | 128 | 129 | 130 | 131 | 132 | 133 | 134 | 135 | 136 | 137 | 138 | 139 | 140 |
| 141 | 142 | 143 | 144 | 145 | 146 | 147 | 148 | 149 | 150 | 151 | 152 | 153 | 154 | 155 | 156 | 157 | 158 | 159 | 160 |
| 161 | 162 | 163 | 164 | 165 | 166 | 167 | 168 | 169 | 170 | 171 | 172 | 173 | 174 | 175 | 176 | 177 | 178 | 179 | 180 |
| 181 | 182 | 183 | 184 | 185 | 186 | 187 | 188 | 189 | 190 | 191 | 192 | 193 | 194 | 195 | 196 | 197 | 198 | 199 | 200 |
| 201 | 202 | 203 | 204 | 205 | 206 | 207 | 208 | 209 | 210 | 211 | 212 | 213 | 214 | 215 | 216 | 217 | 218 | 219 | 220 |
| 221 | 222 | 223 | 224 | 225 | 226 | 227 | 228 | 229 | 230 | 231 | 232 | 233 | 234 | 235 | 236 | 237 | 238 | 239 | 240 |
| 241 | 242 | 243 | 244 | 245 | 246 | 247 | 248 | 249 | 250 | 251 | 252 | 253 | 254 | 255 | 256 | 257 | 258 | 259 | 260 |
| 261 | 262 | 263 | 264 | 265 | 266 | 267 | 268 | 269 | 270 | 271 | 272 | 273 | 274 | 275 | 276 | 277 | 278 | 279 | 280 |
| 281 | 282 | 283 | 284 | 285 | 286 | 287 | 288 | 289 | 290 | 291 | 292 | 293 | 294 | 295 | 296 | 297 | 298 | 299 | 300 |
| 301 | 302 | 303 | 304 | 305 | 306 | 307 | 308 | 309 | 310 | 311 | 312 | 313 | 314 | 315 | 316 | 317 | 318 | 319 | 320 |
| 321 | 322 | 323 | 324 | 325 | 326 | 327 | 328 | 329 | 330 | 331 | 332 | 333 | 334 | 335 | 336 | 337 | 338 | 339 | 340 |
| 341 | 342 | 343 | 344 | 345 | 346 | 347 | 348 | 349 | 350 | 351 | 352 | 353 | 354 | 355 | 356 | 357 | 358 | 359 | 360 |
| 361 | 362 | 363 | 364 | 365 | 366 | 367 | 368 | 369 | 370 | 371 | 372 | 373 | 374 | 375 | 376 | 377 | 378 | 379 | 380 |
| 381 | 382 | 383 | 384 | 385 | 386 | 387 | 388 | 389 | 390 | 391 | 392 | 393 | 394 | 395 | 396 | 397 | 398 | 399 | 400 |
| 401 | 402 | 403 | 404 | 405 | 406 | 407 | 408 | 409 | 410 | 411 | 412 | 413 | 414 | 415 | 416 | 417 | 418 | 419 | 420 |
| 421 | 422 | 423 | 424 | 425 | 426 | 427 | 428 | 429 | 430 | 431 | 432 | 433 | 434 | 435 | 436 | 437 | 438 | 439 | 440 |
| 441 | 442 | 443 | 444 | 445 | 446 | 447 | 448 | 449 | 450 | 451 | 452 | 453 | 454 | 455 | 456 | 457 | 458 | 459 | 460 |
| 461 | 462 | 463 | 464 | 465 | 466 | 467 | 468 | 469 | 470 | 471 | 472 | 473 | 474 | 475 | 476 | 477 | 478 | 479 | 480 |
| 481 | 482 | 483 | 484 | 485 | 486 | 487 | 488 | 489 | 490 | 491 | 492 | 493 | 494 | 495 | 496 | 497 | 498 | 499 | 500 |

8. The gaps between primes tends to get bigger as you go along. Lets see how much bigger this gaps gets.[1]

   (a) Between 1 and ______________ , there are 10 primes.
   (b) Between 100 and ______________ , there are 10 primes.
   (c) Between 400 and ______________ , there are 10 primes.
   (d) The longest run of consecutive composite numbers between 1 and 100 starts with ______________ and ends with ______________ and is ______________ long.
   (e) The longest run of consecutive composite numbers between 401 and 500 starts with ______________ and ends with ______________ and is ______________ long.

9. Do the gaps between the primes get long and longer or is there eventually a record-size gap that never gets beaten?

   Factorials can help understand this problem. A factorial, like 5! means to multiply all the numbers between 1 and 8 or $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$.

   (a) Is the number below prime or composite? (Can you find a factor that is not 1 or itself?)

   $$10! + 2 = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 2$$

   (b) Is the number below prime or composite?

   $$10! + 3 = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 3$$

   (c) Based on your answers above, how many composite numbers must there be in a row, starting with $10! + 2$?
   (d) Name a number which you can be sure is in a string of 99 composite numbers.
   (e) Name a number which you can be sure is in a string of 999 composite numbers.
   (f) Explain how you can be sure any record of composite numbers is a row eventually gets broken.

10. At the beginning of your list there are three odd primes in a row: $3, 5, 7$.

   (a) Does this ever happen again in your list?
   (b) Can it ever happen later on, perhaps beyone 500? Why or why not?

11. You can find pairs of primes in your list that are two odd numbers in a row which are both prime. These are called *twin primes*, such as 17 and 19 or 41 and 43.

   (a) List all the twin primes less than 500.
   (b) We know there are infinitely many primes. We also know there is only one *prime triplet.* How many twin primes are there?

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[1]This problem is alluding to a famous theorem, called the Prime Number Theorem, that states that the number of prime less than or equal to $x$ is approximately equal to $\frac{x}{\ln x}$.

12. See what numbers can be made by adding two primes. For example

$$2+2=4, \quad 2+3=5, \quad 3+3=6, \quad 2+5=4, \quad 3+5=8, \quad 2+7$$

   (a) Make all numbers less than 20 by adding two primes.

   (b) Find a composite number that is not the sum of two primes. Hint: there is one in the 20s.

   Is there such a number is the 30s? 40s? 50s?

   (c) Find a pattern for such numbers. Hint: look at the size of the gaps between primes. Explain, based on your answer, exactly how you can tell whether an odd composite number is the sum of two primes or not.

   (d) Can you find an even number which is not prime, and not the sum of two primes?[2]

13. A perfect square is what you get when you multiply an integer by itself.

   (a) Write down the first 10 perfect sqaures.

   (b) List all the prime numbers less than 100. For each prime, determine if it can be written as a sum of two perfect squares. For example, $5 = 4 + 1$ can be written as a sum but 7 can not.

   For the numbers that can be written as a sum of squares, keep track of the squares you used.

   (c) Find a pattern that tells you which primes can be written as the sum of two squares. Hint: look for where they fall in the sieve.

   (d) Explain why the impossible numbers really are impossible.

   (e) Explain why all the primes that are possible to write as a sum of squares really are possible (why does your pattern you found will work forever?). (Warning: difficult!)

14. **Prime Generating Functions:** Here we want to find a function (or formula) that gives only prime numbers and, ideally, all the prime numbers. Here are two such functions that don't quite work as expected.

[2] This is a famous conjecture called Goldbach's conjecture: *Every even integer n greater than two is the sum of two primes.* A similar statement (actually equivalent to Goldbach's conjecture) is *Every integer n greater than five is the sum of three primes.* The Goldbach conjecture has not been proved or disproved.

(a) Look at one less than powers of two. So, $2^1 - 1 = 1$ is not a prime. $2^2 - 1 = 3$ is prime.

| **n** | $\mathbf{2^n - 1}$ | **Is Prime?** |
|---|---|---|
| 1 | 1 | No |
| 2 | 3 | Yes |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |
| 11 | | |

Finish filling out the table, maybe even do some extras. See if there is a pattern for when $2^n - 1$ is prime and when it is not prime.

(b) Try this formula $n^2 - n + 41$ (due to Euler) and see if generates any primes.

| **n** | $\mathbf{n^2 - n + 41}$ | **Is Prime?** |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |
| 11 | | |

How about negative numbers? How long will this pattern hold?

15. Here is a list of all the primes less than 2500 (graphic thanks to Paul Zeitz).

Prime Numbers to 2500 (+2,5)