

# Problem Solving Methods

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One of the main points of problem solving is to learn techniques by just doing problems. So, lets start with a few problems and learn a few techniques.

## Patience

1. Find a 9-digit number using each digit 1 through 9 once, such that the first  $n$  digits are divisible by  $n$
2. Sheep and Wolves. On a  $5 \times 5$  chessboard place 5 wolves (who move like chess queens) and 3 sheep so that the sheep are safe from being eaten by the wolves.
3. Triangle problem: arrange the numbers 1 through 6 into a “difference triangle” where each number in the row below is the difference of the two numbers above it. For example

$$\begin{array}{ccc} 6 & 4 & 1 \\ & 2 & 3 \\ & & 1 \end{array}$$

almost works but it has two 1's and no 5.

How about with 10 numbers? 15?

## Try special cases (make up an easier problem!)

4. How many zeros are at the end of the number

$$100! = 100 \cdot 99 \cdot 98 \cdot 97 \cdots 3 \cdot 2 \cdot 1$$

5. The numbers 1 through 100 are written on the board. Take two numbers,  $u$  and  $v$  and erase them writing  $uv + u + v$  in their place. This leaves 99 numbers. Keep doing this and after a while, there will only be one number left on the board. What are the possible numbers left?

# Getting dirty

6. What is the smallest number that can not be written by subtracting a prime from a square. For example

$$1 = 4 - 3$$

$$2 = 9 - 7$$

$$3 = ?$$

(How about the next smallest number?)

7. In the year 1971, Smith said, "I was  $n$  years old in the year  $n^2$ ." When was Smith born?
8. For every positive integer  $n$ , look at the number  $n^3 - n$ . The first few are here:

$n$	$n^3 - n$
1	0
2	6

Keep filling out this chart. For at least the first few numbers in the  $n^3 - n$  column, they should be divisible by 3.

- (a) Are all the numbers  $n^3 - n$  divisible by 3?
- (b) If not, find one that is not. If so, show that this is always the case.
9. For every positive integer  $n$ , look at the number  $n^5 - n$ . The first few are here:

$n$	$n^5 - n$
1	0
2	40

Keep filling out this chart. For at least the first few numbers in the  $n^5 - n$  column, they should be divisible by 5.

- (a) Are all the numbers  $n^5 - n$  divisible by 5?
- (b) If not, find one that is not. If so, show that this is always the case.

# Organization

10. What is the greatest number of regions into which three straight lines (of infinite extent) can divide the plane? How about 4 lines? How about  $n$  lines?
11. What do you get if you add up all the numbers from 1 to 100? Can you do this by using any tricks? What if you just add them up directly?

12. Fill out the multiplication table below.

$\times$	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										

What do you get if you add up all the numbers in the table?

13. How many rectangles are in a  $10 \times 10$  rectangle?

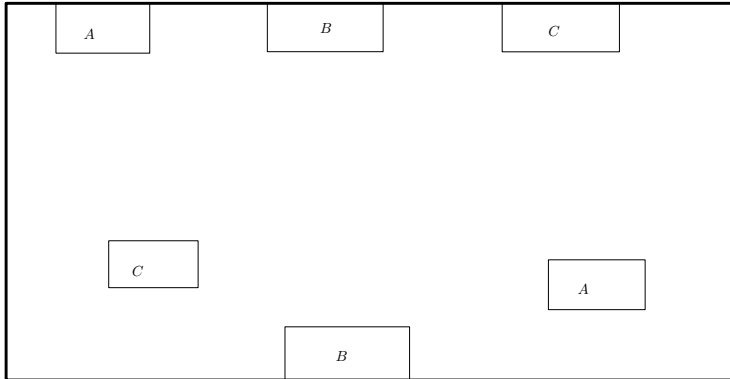
14. Magic Squares: Imagine a  $3 \times 3$  array of squares:


You are to put numbers, 1 through 9, one in each square, so that each row and column add up to the same number.

- What are the possibilities for the sum of the rows and columns?
- What are the possible ways to add three numbers between 1 and 9 to the sum you found in the previous part?
- Find a  $3 \times 3$  magic square.
- Can you turn your  $3 \times 3$  magic square into a better magic square by making sure the diagonals also have the same sum?
- What about  $2 \times 2$  magic square?  $4 \times 4$ ?

# Wishful thinking

15. Connect  $A$  to  $A$ ,  $B$  to  $B$  and  $C$  to  $C$  without crossing lines or leaving the box.



# Symmetry

Many of the problems here can use symmetry. For example, in the triangle problem (Problem 3), it is helpful to realize that the following triangles are essentially the same:

$$\begin{array}{ccc} 6 & 4 & 1 \\ 2 & 3 & \\ 1 & & \end{array} \quad \begin{array}{ccc} 1 & 4 & 6 \\ 3 & 2 & \\ 1 & & \end{array}$$

Another example is the problem of the sheep and the wolves (Problem 2). In this problem, there is a symmetry between the wolves and the sheep. The case with 5 wolves and 3 sheep is equivalent to the problem with 3 wolves and 5 sheep (do you see why?).

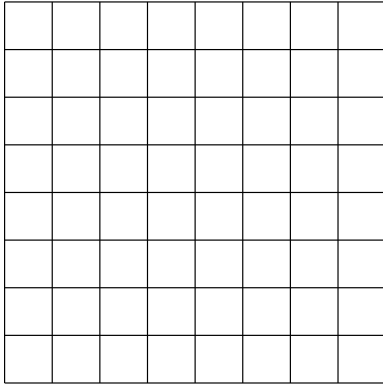
16. In the game of tic-tac-toe, how many different first moves are there?  
For each of the first moves, how many second moves are there?

# Invariants

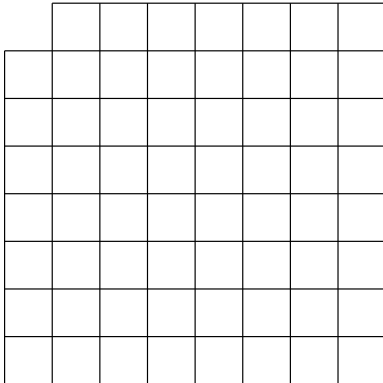
17. Here are a few classic coloring problems.

- (a) Take an  $8 \times 8$  board (containing 64 squares) and cover it with 32 dominoes of size  $2 \times 1$ , so each domino covers 2 adjacent squares.

(Side note: In 1961 the British physicist M.E. Fisher showed that there are 12,988,816 ways to do this.)



- (b) Now take two opposite corners off the board and cover it with 31 dominoes:



18. Let  $n$  be an odd number and write the numbers  $1, 2, \dots, 2n$  on the board. Then, pick any two numbers  $a, b$  and erase them and replace them with  $|a - b|$ . This leaves  $2n - 1$  numbers left on the board. keep doing this until only one number is left.

Do you end up with an odd number or an even number or does it depend?

## Other Problems

19. Imagine a wooden cube with side length of 3 inches. Imagine cutting the cube into 27 smaller cubes of side length 1, using straight cuts (think of a Rubik's cube). What is the minimum number of straight cuts needed to make this happen?

What if it is a  $4 \times 4 \times 4$  cube? What is the minimum number of cuts?  $5 \times 5 \times 5$ ?