

Greatest Common Divisor

Blake Thornton

Much of this is due directly to Josh Zucker, Paul Zeitz and Harold Reiter.

Divisibility

1. How can 48 be built from primes? In what sense is there only one way? How about 234? How can you organize your work to find the prime factors of numbers such as 2008?
2. What are all the divisors of 48? Of 120? Of 2008? How many different ways can you find to organize these answers? Can you tell how many divisors a number will have by looking at its prime factorization?
3. Make a list of the numbers, 1 through 30, and the number of divisors of that number. You may have heard of this called the tau function, $\tau(n)$, the number of divisors of n . What do you notice?

In particular, look for differences when the number of divisors is even or odd. Also, look at the sums of divisors (sometimes denoted $\sigma(n)$). Also, compare the $\tau(n)$, $\tau(m)$ and $\tau(nm)$ for any n, m .

Greatest Common Divisor

The greatest common divisor of two numbers is the largest number that divides evenly into both. For example, the greatest common divisor of 10 and 15 is 5:

$$\gcd(10, 15) = 5$$

4. Compute some greatest common divisors.

$$\gcd(12, 8) =$$

$$\gcd(12, 4) =$$

$$\gcd(36, 84) =$$

$$\gcd(98, 120) =$$

$$\gcd(1232, 24) =$$

$$\gcd(2008, 98) =$$

$$\gcd(4862, 47320) =$$

Explain how you know that you found the greatest common divisor.

Euclidean Algorithm (to find GCD)

Suppose you want to find $\gcd(84, 66)$. First you divide and find the remainder:

$$84 = 66 \cdot 1 + 18$$

Then, notice that $\gcd(84, 66) = \gcd(66, 18)$. The point is that the gcd of two numbers is the same as the gcd of one of the numbers and the remainder (WHY?)

Therefore, to find the gcd we just keep dividing making the problem simpler at each stage:

$$84 = 66 \cdot 1 + 18$$

$$66 = 18 \cdot 3 + 12$$

$$18 = 12 \cdot 1 + 6$$

$$12 = 6 \cdot 2$$

and therefore $\gcd(84, 18) = 6$.

5. Use the Euclidean Algorithm to find the greatest common divisor for the numbers in Problem 4.
6. Determine if the following statements are true or false:
 - (a) If $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$ then $\gcd(b, c) = 1$.
 - (b) If $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$ then $\gcd(a, bc) = 1$.
 - (c) If $\gcd(a, b) = 2$ and $\gcd(a, c) = 2$ then $\gcd(b, c) = 2$.
 - (d) If $\tau(a) = n$ and $\tau(b) = m$ then $\tau(ab) = mn$ (see Problem 3).
7. **Least Common Multiple.** The *least common multiple* of two numbers is the smallest number that is a multiple of each.

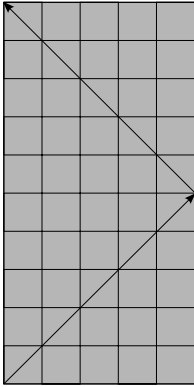
Make a chart of a bunch of numbers, their greatest common divisor and least common multiple. I've started your chart for you. Write each of your numbers in its prime factorization (I have not done this for you).

| a | b | $\gcd(a, b)$ | $\text{lcm}(a, b)$ |
|-----|-----|--------------|--------------------|
| 2 | 3 | | |
| 4 | 6 | | |
| 12 | 14 | | |
| 10 | 15 | | |
| 2 | 4 | | |
| 7 | 21 | | |
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- (a) What is $12 \cdot 18$? How much is $\gcd(12, 18)$ times the least common multiple of 12 and 18?
- (b) Find a relationship between $a, b, \gcd(a, b)$, and $\text{lcm}(a, b)$.
- (c) What is the smallest positive integer that is divisible by each of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10?

8. **Pool Table:** Imagine a pool table with pockets only in the corners (represent it in graph paper). Start with a ball at the bottom left corner, moving up at an angle of 45° . The ball bounces off each side of the rectangle until finally it reaches one of the corners and goes in the pocket.

For example, in a 5×10 (width 5, height 10), the ball moves as below:



When the ball goes into a pocket, we will count that as a bounce on the top/bottom *and* the sides of the pool table. So, for the 5×10 table, the ball bounces on the sides 2 times (one real bounce and one bounce for going in the pocket) and on the top/bottom 2 times (one real bounce and one bounce for going in the pocket).

- (a) Draw the situation for a 6×10 table. How many times does the ball bounce on the top/bottom? How many times does the ball bounce on the sides? Which pocket does the ball land in?
- (b) Draw a table of various sizes and make a chart listing the dimensions of the table, how many times the ball bounces in total, how many of those bounces are on the left/right sides, how many of those bounces are on the top/bottom sides, and which pocket the ball lands in. Be sure to include at least the following table sizes (and add more tables too!):

| (Width, Length) | Side Bounces | Top/Bottom Bounces | Pocket |
|-----------------|--------------|--------------------|--------|
| (1, 3) | | | |
| (1, 4) | | | |
| (1, 6) | | | |
| (3, 5) | | | |
| (6, 10) | | | |
| (9, 15) | | | |
| (1, 12) | | | |
| (2, 12) | | | |
| (3, 12) | | | |
| (4, 12) | | | |
| (5, 12) | | | |
| (6, 12) | | | |
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- (c) Think about greatest common divisor and your table. Why do some of the rows in your table give the same answers?
- (d) Predict the correct numbers for the table:

| (Width, Length) | Side Bounces | Top/Bottom Bounces | Pocket |
|-----------------|--------------|--------------------|--------|
| (79, 103) | | | |
| (79, 102) | | | |
| (78, 102) | | | |

Explain your prediction. Prove that you are right.

- (e) Can you construct a table for which the pool ball bounces on the sides exactly 4 times and on the tops and bottom exactly 8 times?
- How about bouncing an arbitrary number of times on the sides and an arbitrary number of times on the top/bottom?

9. **Decanting Problem** (Thanks to Harold Reiter). This is a liquid measuring problem that begins with two ungraduated decanters whose integer capacities a and b are given. The problem is to determine the smallest amount of liquid that can be measured and how such amount can be obtained, by a process of filling, pouring, and dumping. Specifically, there are three actions we can take:

- fill an empty decanter,
- dump out a full decanter, and
- pour from one decanter to the other until either the receiving decanter is full or the poured decanter is empty.

- (a) Let $a = 3$ and $b = 5$. See if you can get 1 unit of water.
- (b) Can you find other ways to get 1 unit using decanters of volume 2 and 3?
- (c) Next, suppose the decanters have capacities 5 units and 7 units. How can you make 1 unit of water now?
- (d) Find the smallest amount of liquid that you can get using decanters of size 15 and 99. Find a way to get this smallest amount.
- (e) Use the Euclidean Algorithm to find $\gcd(257, 341)$. Use your work to solve the decanting problem for decanters of size 257 and 341.
- (f) Use the Euclidean Algorithm to find $\gcd(102, 78)$. Use your work to solve the decanting problem for decanters.

10. **Perfect Numbers.** A number is *perfect* if it is equal to the sum of its proper divisors. The smallest perfect numbers are 6, 28 and 496.

- (a) Write down the prime factorizations of these perfect numbers.
- (b) Check that these numbers really are perfect.
- (c) Find a pattern of the perfect numbers. See if you can guess the next perfect number.
- (d) Prove that the only even perfect numbers are in this pattern (difficult).
- (e) Find an odd perfect number (this is a famous unsolved problem).