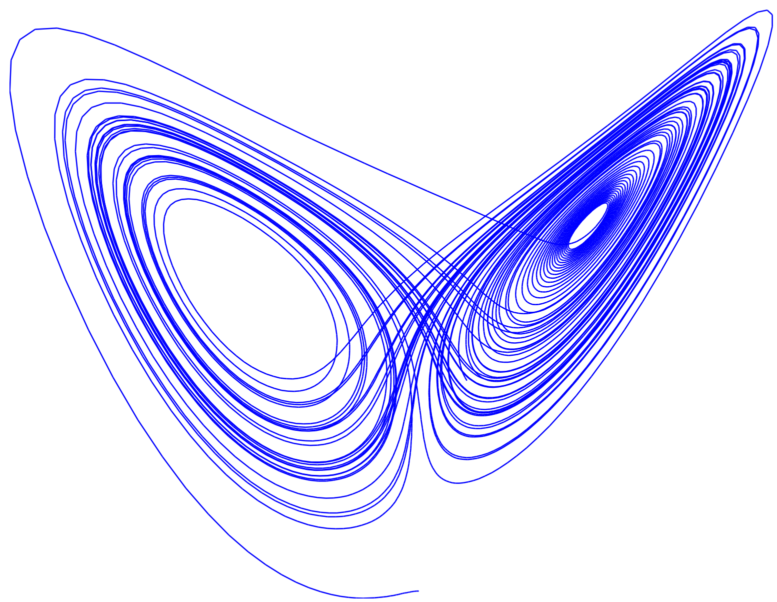
Washington University Teacher Circle

What is Chaos?



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Basic Notions:

The term *chaos* is often used to describe the behavior of deterministic systems which are highly sensitive to perturbations of the initial state of the system.

- A system is *deterministic* if its state an any future time can be determined from complete knowledge of its present state.
- Informally, a deterministic system is *sensitive* to perturbations of the initial state of the system if a small change in the initial state leads to large changes in future states of the system.

Reference: *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos* by Clark Robinson [3].

Differential Equations:

As an example, consider the following differential equation,

$$\frac{dx}{dt} = kx,$$

which models exponential growth. Its solutions can be found through integration:

$$\int \frac{1}{x}\frac{dx}{dt}\, dt = \int k\, dt \Longrightarrow \ln|x(t)| = kt + C \Longrightarrow x(t) = A\exp{(kt)}.$$

This system is, therefore, deterministic since $x(t)$ can be computed for any future time given the value of $A = x(0)$.

Example: Lorenz Equations

- Simplified meteorological system (Edward Lorenz 1960s):

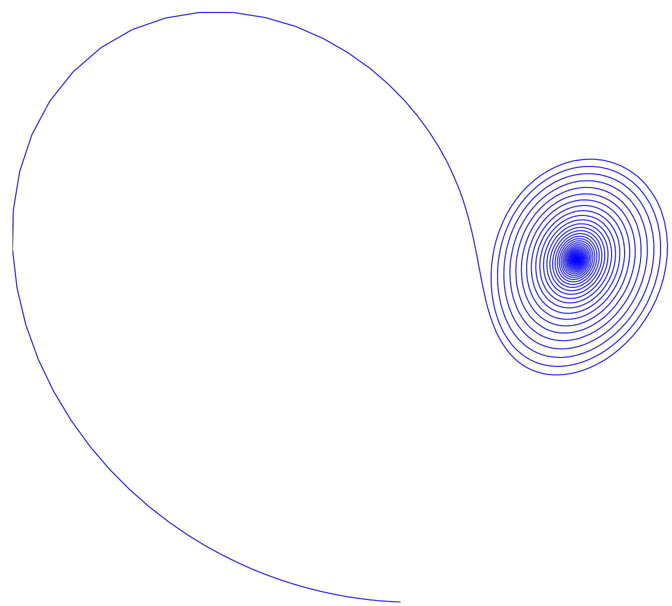
$$\begin{pmatrix} \frac{dX}{d\tau} \\ \frac{dY}{d\tau} \\ \frac{dZ}{d\tau} \end{pmatrix} = \begin{pmatrix} -\sigma X + \sigma Y \\ -XZ + rX - Y \\ XY - bZ \end{pmatrix}, \tag{1}$$

  where σ, r, and b are positive constants.

- As with the exponential model, this system is deterministic because standard results in the theory of differential equations can be applied to guarantee the existence and uniqueness of solutions for given initial conditions.

Example: Lorenz Equations

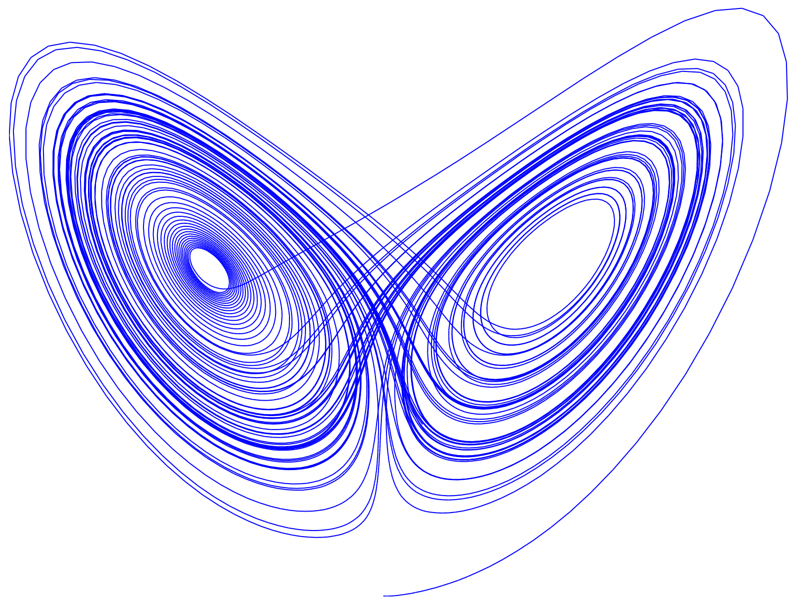
- Orbit with $\sigma = 10$, $b = \frac{8}{3}$, & $r = 20$: (numerical results)



- Solution approaches a stable equilibrium point. Small changes in initial condition will not alter the solution for large times.

Example: Lorenz Equations

- Orbit with $\sigma = 10$, $b = \frac{8}{3}$, & $r = 28$: (numerical results)

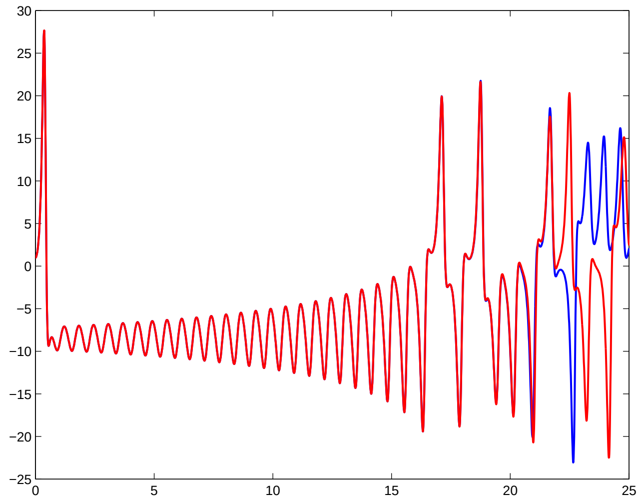


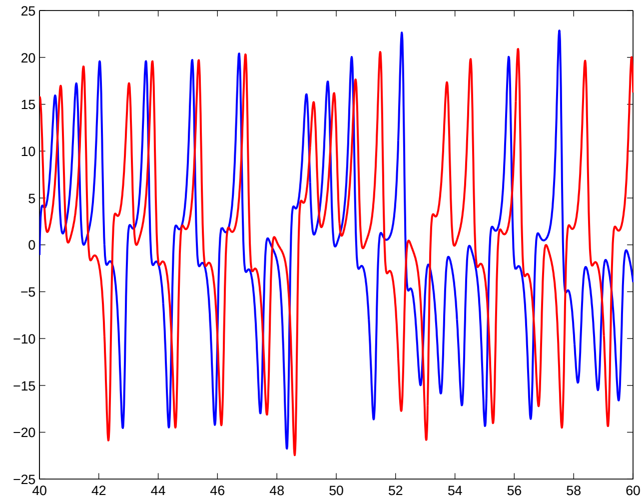
- Solution exhibits chaotic behavior alternatively approaching two separate unstable equilibrium points. Small changes in initial state will likely result in similarly chaotic solutions, but the values at fixed points in time could be highly sensitive to these changes.

Example: Lorenz Equations

- Sensitivity to initial conditions:

$$(X, Y, Z) = (0, 1, 0) \quad \text{versus} \quad (X, Y, Z) = (0, 0.99, 0)$$





- The solutions start out very close, but for large times there is little correlation between the two trajectories. (The graphs depict the Y coordinate.)

The Butterfly Effect

The preceding example motivated Lorenz to coin the term *butterfly effect* to illustrate this phenomenon of sensitivity to initial conditions. He explained that a butterfly flapping its wings in Beijing might affect the weather in the United States a few days later [2].

It is somewhat paradoxical that long-term predictions are essentially impossible with deterministic systems. The obstacle is determining the precise initial state of such systems – any slight error in initial data will eventually lead to inaccurate future predictions.

A similar concept was used as the basis for an interesting short-story, *A Sound of Thunder*, written by Ray Bradbury and published by Collier's magazine in 1952 [1].

Discrete systems

Recall the exponential model introduced earlier, which led to

$$x(t) = x(0)\exp(kt).$$

- For a fixed time step Δt the growth of $x(t)$ obeys:

$$\frac{x(t+\Delta t)}{x(t)} = \frac{x(0)\exp(k(t+\Delta t))}{x(0)\exp(kt)} = \exp(k\Delta t).$$

- In other words, if $t_k = k\Delta t$ and $x_k = x(t_k)$, then

$$x_{k+1} = \alpha x_k,$$

  where $\alpha = \exp(k\Delta t)$.

One can thus consider discrete analogs to various differential equations with the goal of examining chaotic behavior.

Example: Tent Map

Let $f : [0,1] \to [0,1]$ be given by

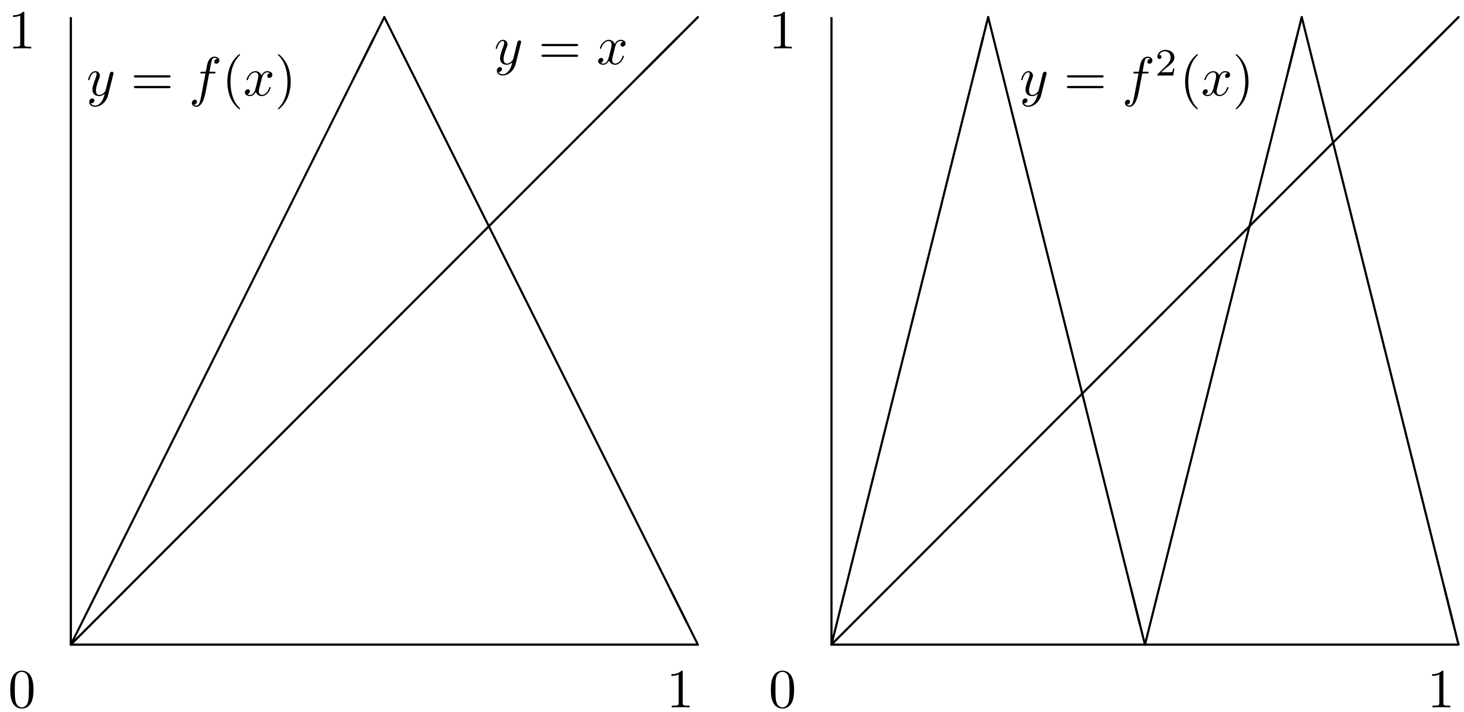
$$f(x) = \begin{cases} 2x, & 0 < x \leq \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} < x \leq 1. \end{cases}$$

We will study the behavior of the points $x_k = f^k(x)$ obtained by composing f with itself k times and evaluating at x. This can be interpreted recursively as in the exponential case,

$$x_k = f^k(x) = f(f^{k-1}(x)) = f(x_{k-1}).$$

We say that x_0 is a *fixed point* of $f(x)$ if $f(x_0) = x_0$. The fixed points of the functions $f^k(x)$ play an important role in the dynamics generated by a given function $f(x)$.

Example: Tent Map



Observe that $x = \frac{2}{3}$ is a fixed point of f. The points $x = \frac{2}{5}, \frac{4}{5}$ are fixed points of f^2 and, in fact, $f(\frac{2}{5}) = \frac{4}{5}$ and $f(\frac{4}{5}) = \frac{2}{5}$.

The points $x = \frac{2}{5}$ and $x = \frac{4}{5}$ form a *cycle* of length 2.

Example: Tent Map

Question: Will the orbit of a point near $x = \frac{2}{3}$ remain nearby?

Consider the orbit of $x = 0.66$:

1	2	3	4	5	6	7	8	9	10	11	12
0.68	0.64	0.72	0.56	0.88	0.24	0.48	0.96	0.08	0.16	0.32	0.64

After two steps, the orbit enters a cycle of length 10. It comes close to $\frac{2}{3}$ at times, but it will indefinitely pass through points far from $\frac{2}{3}$.

How can one mathematically characterize the behavior of orbits for x closer and closer to the fixed point $\frac{2}{3}$?

Example: Tent Map

Let x be a point near $\frac{2}{3}$. Consider the ratio

$$\frac{\text{Distance between } f(x) \text{ and } \frac{2}{3}}{\text{Distance between } x \text{ and } \frac{2}{3}} = \frac{|f(x) - \frac{2}{3}|}{|x - \frac{2}{3}|},$$

which, as x approaches $\frac{2}{3}$, becomes

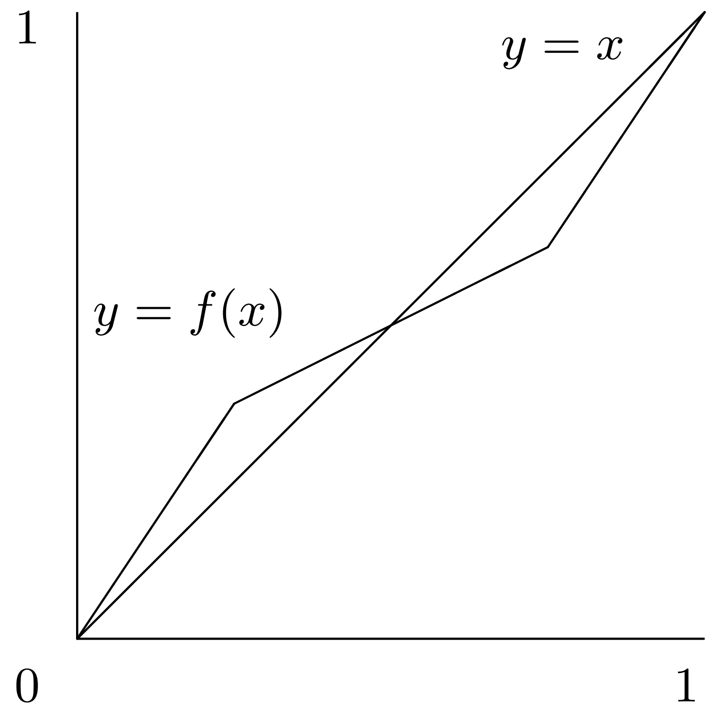
$$\lim_{x \to \frac{2}{3}} \frac{|f(x) - \frac{2}{3}|}{|x - \frac{2}{3}|} = |f'(\frac{2}{3})| = |-2| = 2 > 1.$$

This says that for points very close to $\frac{2}{3}$, $f(x)$ will be twice as far away from $\frac{2}{3}$ as x itself. Whenever this ratio tends to a value strictly greater than one, the fixed point is said to be *repelling*. If, on the other hand, the ratio tends to a value strictly less than one, the fixed point is said to be *attracting*.

An attractor:

Consider the map f given by

$$f(x) = \begin{cases} \frac{3}{2}x, & 0 \le x \le \frac{1}{4}, \\ \frac{1}{2}(x + \frac{1}{2}), & \frac{1}{4} \le x \le \frac{3}{4}, \\ \frac{3}{2}(x - \frac{1}{3}). \end{cases}$$



References

[1] Ray Bradbury. A sound of thunder. *Collier's*, 1952.

[2] Edward N. Lorenz. Deterministic nonperiodic flow. *J. Atmos. Sci.*, 20:130–141, 1963.

[3] Clark Robinson. *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*. CRC Press, 1995.