Math Circle: Perfect Numbers

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The Greek letter $\sigma$ is called sigma.
If $n$ is an integer

$$\sigma(n) = \text{sum of the divisors of } n.$$

Examples:

$\sigma(1) = 1,$
$\sigma(2) = 1 + 2 = 3,$
$\sigma(3) = 1 + 3 = 4,$
$\sigma(4) = 1 + 2 + 4 = 7,$
$\sigma(5) = 1 + 5 = 6,$
$\sigma(6) = 1 + 2 + 3 + 6 = 12.$
1. Compute:
\[ \sigma(1) = \]
\[ \sigma(2) = \]
\[ \sigma(3) = \]
\[ \sigma(4) = \]
\[ \sigma(5) = \]
\[ \sigma(6) = \]
\[ \sigma(7) = \]
\[ \sigma(8) = \]
\[ \sigma(9) = \]
\[ \sigma(10) = \]
\[ \sigma(11) = \]
\[ \sigma(12) = \]
\[ \sigma(13) = \]
\[ \sigma(14) = \]
\[ \sigma(15) = \]
\[ \sigma(16) = \]
\[ \sigma(17) = \]
\[ \sigma(18) = \]
\[ \sigma(19) = \]
\[ \sigma(20) = \]
\[ \sigma(21) = \]
\[ \sigma(22) = \]
\[ \sigma(23) = \]
\[ \sigma(24) = \]
\[ \sigma(25) = \]
\[ \sigma(26) = \]
\[ \sigma(27) = \]
\[ \sigma(28) = \]
\[ \sigma(29) = \]
\[ \sigma(30) = \]
\[ \sigma(31) = \]
\[ \sigma(32) = \]
2. A number $n$ is called perfect if $\sigma(n) = 2n$.
Example: $\sigma(6) = 12$. We have that 6 is a perfect number.
Find more perfect numbers.
3. Find numbers $n$ so that $\sigma(n) > 2n$. 
4. If $p$ is a prime number, then what is $\sigma(p)$? What are $\sigma(p^2)$, $\sigma(p^3)$?

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\sigma(2)$</th>
<th>$\sigma(2^2)$</th>
<th>$\sigma(2^3)$</th>
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<td>2</td>
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<td>$\sigma(2^3) =$</td>
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<td>$\sigma(29) =$</td>
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<td>31</td>
<td>$\sigma(31) =$</td>
<td>$\sigma(31^2) =$</td>
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</tbody>
</table>
5. Compute $\sigma$(powers of 2).

$\sigma(2^1) =$
$\sigma(2^2) =$
$\sigma(2^3) =$
$\sigma(2^4) =$
$\sigma(2^5) =$
$\sigma(2^6) =$
$\sigma(2^7) =$
$\sigma(2^8) =$
6. Compare $\sigma(a)\sigma(b)$ to $\sigma(ab)$:

$\sigma(2)\sigma(3) = \sigma(6) = \sigma(2)$
$\sigma(2)\sigma(5) = \sigma(10) = \sigma(2)$
$\sigma(2)\sigma(7) = \sigma(14) = \sigma(2)$
$\sigma(2)\sigma(13) = \sigma(26) = \sigma(2)$
$\sigma(4)\sigma(3) = \sigma(12) = \sigma(2)$
$\sigma(4)\sigma(5) = \sigma(20) = \sigma(2)$

$\sigma(2)\sigma(2) = \sigma(4) = \sigma(2)$
$\sigma(2)\sigma(4) = \sigma(8) = \sigma(2)$
$\sigma(2)\sigma(6) = \sigma(12) = \sigma(2)$
$\sigma(4)\sigma(6) = \sigma(24) = \sigma(2)$
$\sigma(3)\sigma(3) = \sigma(9) = \sigma(2)$
$\sigma(3)\sigma(9) = \sigma(27) = \sigma(2)$
$\sigma(3)\sigma(6) = \sigma(18) = \sigma(2)$
**Homework:** 1. Find more perfect numbers \((\sigma(n) = 2n)\).
2. Give more examples of numbers \(n\) such that \(\sigma(n) > 2n\).
3. Give examples of numbers \(n\) such that \(\sigma(n) > 3n\).
4. Is it true that \(\sigma(n)\) is always less than \(4n\)?