Catalan Numbers

These numbers arise in many mathematical situations. We'll explore 4 of them. We'll investigate:

1) Diagonal Avoiding Paths
2) Polygon Triangulation
3) Binary Trees
4) Polyominoes

Let's begin by exploring...
1) In a grid of $n \times n$ squares, how many paths are there of length $2n$ that lead from the upper left corner to the lower right corner that do not touch the diagonal dotted line from upper left to lower right? In other words, how many paths stay on or above the main diagonal?
Use the squares on the following two pages to complete the following table.

<table>
<thead>
<tr>
<th>n</th>
<th>$A_n$ = # of above mentioned paths</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A_1$ =</td>
</tr>
<tr>
<td>2</td>
<td>$A_2$ =</td>
</tr>
<tr>
<td>3</td>
<td>$A_3$ =</td>
</tr>
<tr>
<td>4</td>
<td>$A_4$ =</td>
</tr>
</tbody>
</table>
$\eta = 1$

$\eta = 2$

$\eta = 3$
$\eta = 4$
(2) Polygon Triangulation

Count the number of ways to triangulate a regular polygon with \( n+2 \) sides.

Examples

\[ n=1 \quad n+2=3 \]

\[ \triangle \quad \text{already triangulated} \]

\[ n=2 \quad n+2=4 \]

\[ \square \quad \text{This diagonal line triangulates the square.} \]

\[ n=3 \quad n+2=5 \]

\[ \text{Triangulated Pentagon} \]

\[ n=4 \quad \text{Triangulated Hexagon} \]
Complete the following table using the shapes on the next 2 pages.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n+2$</th>
<th>$B_n$ = # of triangulations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>$B_1$ =</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$B_2$ =</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$B_3$ =</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$B_4$ =</td>
</tr>
</tbody>
</table>

By convention, we take $B_0 = 1$. 
$\ell = 4$
(3) Binary Trees

On the next page are examples of binary trees with \( n \) internal nodes for \( n = 0, 1, 2, 3, \) and \( 4. \)

A rooted binary tree is an arrangement of points (nodes) and lines connecting them where there is a special node (the root) and as you descend from the root, there are either two lines going down or zero. Internal nodes are the ones that connect to two nodes below.
Using the above examples, complete the following table to find the number of binary trees for each $n$. 
<table>
<thead>
<tr>
<th>n</th>
<th>( C_n ) = # of binary trees with ( n ) internal nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
(4) Skew Polyominoes

A polyomino is a set of squares connected by their edges. A skew polyomino is a polyomino where the bottom of the column to the left is always lower or equal to the bottom of the column to the right. Similarly, the top of the column to the left is lower than or equal to the top of the column to the right.
For each $n$, let $D_n$ denote the number of skew polyominoes that have perimeter $2n+2$. Note that it is the perimeter that is fixed and not the number of blocks. Using the examples on the next page as motivation, complete the following table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$2n+2$</th>
<th>$D_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: $D_0 = 1$ by convention.
Polyomino Examples

\( \eta = 1 \)  

\( \eta = 2 \)  

\( \eta = 3 \)  

\( \eta = 4 \)
**General Counting Formulas**

For the 4 examples, we have counted the number of respective objects for \( n=0, n=1, n=2, n=3, \) and \( n=4. \)

However, brute force cannot work for \( n=k, \) where \( k \) is any integer because the numbers simply get too big & the diagrams get too confusing.

Instead, you will develop a **recursive relationship** that allows us to calculate the number of objects for \( n \geq 5. \)

**Definition:** A **recursive relationship** is a formula for a sequence of numbers: \( c_0, c_1, c_2, c_3, \ldots, c_n, \ldots \)

That gives the value for \( c_n \) in terms of all of the previous numbers \( (c_0, \ldots, c_{n-1}). \)

**Example:** The formula:

\[
\begin{align*}
c_0 &= 1 \\
c_1 &= 2 \\
c_n &= c_{n-1} + c_{n-2}
\end{align*}
\]

is a recursive relationship for the following sequence of numbers:

\( 1, 2, 3, 5, 8, 13, 21, \ldots \)

Now write down another example of a recursive relationship:
Recursively Counting Polygon Triangulations

Now we will develop a recursive relationship that will allow us to calculate the # of polygon triangulations for \( n \geq 5 \).

Let \( P_n \) denote the number of ways to triangulate a regular polygon with \( n+2 \) sides.

You already have \( P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, \) & \( P_4 = 14 \).

We want to write \( P_n \) so that it just depends on \( P_0, P_1, \ldots P_{n-1} \).

Let's first consider \( n = 6 \) so that we are triangulating an octagon.

Consider the shaded line.

The shaded line has to be part of exactly 1 triangle. How many possibilities are there?
Now fix a triangle which has the shaded line as a side.

There is a polygon on the right of the triangle & one on the left. How many ways can each of them be triangulated? (just use the notation P1,P2,...,P5)

__________________________

Leaving the initial triangle fixed, how many ways can the octagon (as pictured above) be triangulated? __________

Now consider the other possible triangles the shaded line could have been a part of.

For each, determine how many ways the octagon, with that initial triangle, can be triangulated.
Add up those possibilities to get the total # of ways an octagon can be triangulated:

\[ P_6 = \]

Now we have a recursive relationship for \( P_6 \).

Is there a generalization of this formula for \( P_n \) with \( n \neq 6 \)?

Explain why the above formula works:

Verify that it works in the \( n=1, n=2, n=3, \) & \( n=4 \) cases.
Show that the same recursive relationship that holds for triangulation of polygons also holds for binary trees.
Can you explain why the same recursion relation holds for non-diagonal crossing paths and skew polyominoes?
An explicit formula for the \( C_n \).

A polynomial is a function of the form

\[ f(x) = a_0 + a_1 x + \cdots + a_n x^n. \]

**Examples**

(1) \( 1 + 3x + 5x^2 + x^3 \)

(2) \( 3 + 87x^{15} + x^{100} \)

are both polynomials.
We can multiply polynomials and use the distributive property:

\[(1 + 3x + x^2)(2 + x + 2x^2)\]

\[= (1)(2 + x + 2x^2)\]
\[+ 3x(2 + x + 2x^2)\]
\[+ x^2(2 + x + 2x^2)\]

\[= 2 + x + 2x^2\]
\[+ 6x + 3x^2 + 6x^3\]
\[+ 2x^2 + x^3 + 2x^4\]

\[= 2 + 7x + 7x^2 + 7x^3 + 2x^4\]
Multiply

\((1 + 2x + 3x^2 + 4x^3)(x + 3x^3 + 7x^5)\)
A **power series** is a polynomial with infinitely many terms.

**Example**

\[ 1 + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots \]

\[ = \sum_{n=0}^{\infty} x^n \]

and

**Example**

\[ x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{5} x^5 + \ldots \]

\[ = \sum_{n=1}^{\infty} \frac{1}{n} x^n \]

are both power series.
In general, a power series is written like

\[ \sum_{n=0}^{\infty} a_n x^n \]

where \( a_n \) is some sequence.

Just like polynomials, we can multiply power series together.
\[(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots) \cdot (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots)\]

\[= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) x^3 + \ldots\]

Use the above formula to multiply the two power series examples together.
Now let $C_n$ denote the Catalan sequence and consider the power series

$$
\sum_{n=0}^{\infty} C_n \cdot x^n
$$

$$= 1 + x + 2x^2 + 5x^3 + 14x^4 + \ldots
$$

Multiply out

$$\left(\sum_{n=0}^{\infty} C_n \cdot x^n\right)\left(\sum_{n=0}^{\infty} C_n \cdot x^n\right)$$
\[(c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots) \cdot (c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots)\]

What do you get?
By using the recursive relation, you should get

\[ c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \ldots \]

which is almost

\[ c_0 + c_1 x + c_2 x^2 + \ldots \]

multiply

\[ c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \ldots \]

through by \( x \) and then add \( c_0 \) to the
result. What do you get?
If we write
\[ f(x) = \sum_{n=0}^{\infty} C_n x^n, \]
we see
\[ [f(x)]^2 = C_0 + x[f(x)]^2 \]
or
\[ x[f(x)]^2 - f(x) + 1 = 0 \]

Apply the Quadratic formula to the above equation. What do you get?
Only consider the + sign and we get

\[ f(x) = \frac{1 - \sqrt{1-4x}}{2x} \]

It can be shown that

\[ \frac{1 - \sqrt{1-4x}}{2x} = 1 + \frac{1}{2!} 2x + \frac{3.1}{3!} 4x^2 + \frac{5.3.1}{4!} 8x^3 + \frac{7.5.3.1}{5!} 16x^4 + \ldots \]
what is the formula for the power series on the previous page?
The coefficients are the Catalan Numbers.