Nontransitive dice

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1. Consider the spinner

![Spinner Diagram]

The base of the spinner is fixed but the arrows (with letter labels) can spin. Your opponent picks an arrow (A, B, or C) and then you pick one of the two remaining arrows.

What is your strategy to win?

Is there a “best” arrow to choose if you are the first player?

**Solution:**

\[ A > B > C > A \]

2. You will find several sets of dice in the room. Don’t mix up the dice in the different sets.

Your goal is to determine which dice in each set will “win” against the other dice. (Two dice are thrown and the winning dice is the larger number.)

Here are what the dice look like:

![Dice Set A]

Figure 1: Dice Set A
Figure 2: Dice Set B (Effron’s Dice)

Figure 3: Dice Set C

Figure 4: Dice Set D (Minwin’s Dice)

(a) For Dice Set A, battle off each pair of dice at least 20 times for each pair. Record the number of wins.

<table>
<thead>
<tr>
<th></th>
<th>Dice 1 wins</th>
<th>Dice 2 wins</th>
<th>Dice 3 wins</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dice 1 vs 2</td>
<td>5/9</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Dice 1 vs 3</td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Dice 2 vs 3</td>
<td></td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>

(b) For Dice Set B, battle off each pair of dice at least 20 times for each pair. Record the number of wins.

<table>
<thead>
<tr>
<th></th>
<th>Dice 1 wins</th>
<th>Dice 2 wins</th>
<th>Dice 3 wins</th>
<th>Dice 4 wins</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dice 1 vs 2</td>
<td></td>
<td>X</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Dice 1 vs 3</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Dice 1 vs 4</td>
<td></td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dice 2 vs 3</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Dice 2 vs 4</td>
<td></td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dice 3 vs 4</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>

(c) For Dice Set C, battle off each pair of dice at least 20 times for each pair. Record the number of wins.
(d) For Dice Set D, battle off each pair of dice at least 20 times for each pair. Record the number of wins.

<table>
<thead>
<tr>
<th></th>
<th>Dice 1 wins</th>
<th>Dice 2 wins</th>
<th>Dice 3 wins</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dice 1 vs 2</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Dice 1 vs 3</td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Dice 1 vs 4</td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Dice 2 vs 3</td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dice 2 vs 4</td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Dice 3 vs 4</td>
<td></td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>

(e) For each of the sets of dice, determine the probability of victory for each pair of dice in the set.

**Solution:**

\[
P(A_1 > A_2) = P(A_2 > A_3) = P(A_3 > A_1) = \frac{5}{9}
\]
\[
P(B_1 > B_2) = P(B_2 > B_3) = P(B_3 > B_4) = P(B_4 > B_1) = \frac{2}{3}
\]
\[
P(B_1 > B_3) = \frac{4}{9}
\]
\[
P(B_2 > B_1) = \frac{1}{3}
\]
\[
P(B_2 > B_4) = \frac{1}{2}
\]
\[
P(B_3 > B_2) = \frac{1}{3}
\]
\[
P(B_3 > B_4) = \frac{1}{3}
\]
\[
P(B_4 > B_3) = \frac{1}{3}
\]
\[
P(B_4 > B_2) = \frac{1}{2}
\]
\[
P(D_1 > D_2) = \frac{1}{12}
\]
\[
P(D_1 > D_3) = \frac{1}{12}
\]
\[
P(D_2 > D_3) = \frac{1}{12}
\]
\[
P(D_3 > D_1) = \frac{1}{12}
\]

(f) Here is the game. You start with one of the above sets of dice. Your opponent gets to select a dice first and then you select a dice. You both roll and the winner has the high dice.

Describe the best way to play this game.
3. Knuth’s Bingo Cards: Numbers from 1 thru 6 shall be called out in a random permutation. A Bingo card shall have two rows of two numbers each. The first player to get both numbers in any row wins. Here are the Bingo Cards:

\[
\begin{array}{cc}
A & B \\
\hline
1 & 2 \\
3 & 4 \\
\end{array}
\]

\[
\begin{array}{cc}
B & C \\
\hline
2 & 4 \\
5 & 6 \\
\end{array}
\]

\[
\begin{array}{cc}
C & D \\
\hline
1 & 3 \\
4 & 5 \\
\end{array}
\]

\[
\begin{array}{cc}
D & \\
\hline
1 & 5 \\
2 & 6 \\
\end{array}
\]

Which cards beat the other cards? (Which card do you want?)

**Solution:** A beats B who beats C who beats D who beats A!

4. The above examples are of *nontransitive* relations. For example, in Dice Set A we have Dice 1 beats Dice 2 beats Dice 3 beats Dice 1.

There are many *transitive* relations in mathematics (and the world). For example with number, “greater than” is transitive as in

\[
\text{If } A > B \text{ and } B > C \text{ then } A > C. 
\]

(a) Think of more transitive relations.

**Solution:** Some examples are “heavier than”, “taller than” and “inside”.

(b) Think of some non-transitive relations.

**Solution:** Some examples are “father of” and “rock-paper-sissors.”

5. Dice construction challenge.

You are to construct three dice to be used in the game described in Problem 2f. Here are the restrictions on your dice:

- Each dice must have three distinct numbers between 1 and 9.
- Pairs of opposite faces must be identical.
- You want to always be able to win (in other words, no matter which dice your opponent chooses, one of two remaining dice throws a number larger than your opponent, on average).

**Solution:** The two solutions corresponds to the rows and columns of the magic square

\[
\begin{array}{ccc}
8 & 1 & 6 \\
3 & 5 & 7 \\
4 & 9 & 2 \\
\end{array}
\]
6. Penny’s Game.

The idea of this game is the following. You will flip a coin until until the sequence of a given length choosen by either you or your opponent comes up, and that person wins. Your opponent chooses first.

For example, if you are playing the length 2 game, perhaps you opponent chooses “HT” and you choose “TH”. Then, you start flipping a penny until one of these two sequences occurs. For example, if you flip “HHHT”, your opponent wins.

(a) What are all possible sequences to select in the length 2 game?

**Solution:** HH, HT, TT, TH

(b) If your opponent wisely selects a length 2 sequence, can you always (or never) select a length 2 sequence that should beat him?

(c) What are all the possible sequences to select in the length 3 game?

**Solution:** HHH, HHT, HTH, THH, HTT, THT, TTH, TTT

(d) If your opponent wisely selects a length 3 sequence, can you always (or never) select a length 3 sequence that should beat him?

**Solution:**

<table>
<thead>
<tr>
<th>1st Player’s choice</th>
<th>2nd Player’s choice</th>
<th>Odds in favor of 2nd Player</th>
</tr>
</thead>
<tbody>
<tr>
<td>HHH</td>
<td>THH</td>
<td>7 to 1</td>
</tr>
<tr>
<td>HHT</td>
<td>THH</td>
<td>3 to 1</td>
</tr>
<tr>
<td>HTH</td>
<td>HHT</td>
<td>2 to 1</td>
</tr>
<tr>
<td>HTT</td>
<td>HHT</td>
<td>2 to 1</td>
</tr>
<tr>
<td>THH</td>
<td>TTH</td>
<td>2 to 1</td>
</tr>
<tr>
<td>THT</td>
<td>TTH</td>
<td>2 to 1</td>
</tr>
<tr>
<td>TTH</td>
<td>HTT</td>
<td>3 to 1</td>
</tr>
<tr>
<td>TTT</td>
<td>HTT</td>
<td>7 to 1</td>
</tr>
</tbody>
</table>

7. Other dice
If you roll two “normal” six sided dice and add the numbers together, what are the probability of the outcomes?

This is worthwhile doing by experiment: start rolling dice and start recording how many times you get each roll.

<table>
<thead>
<tr>
<th>Roll</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

Now, try the same thing with the Sicherman Dice below: (To do this you will need to first determine what the possible sums of both dice are.)

![Figure 7: Dice Set S (Sicherman Dice)](image)

determine what the possible sums of both dice are.)