Math 331: Homework 7, Due Oct 21

1. Using only techniques discussed in class and/or the text, factor completely over $\mathbb{C}$:
   (a) $x^5 - 19x^4 + 139x^3 - 485x^2 + 800x - 500 = (x-5)^3(x-2)^2$
   (b) $x^6 - 12\,x^5 + 58\,x^4 - 144\,x^3 + 193\,x^2 - 132\,x + 36 = (x-3)^2\,(x-2)^2\,(x-1)^2$
   (c) $x^4 - 14\,x^3 + 71\,x^2 - 154\,x + 120 = (x-5)\,(x-4)\,(x-3)\,(x-2)$
   (d) $x^5 - 5\,x^4 + 7\,x^3 - 2\,x^2 + 4\,x - 8 = (x-2)^3\,\left(x^2 + x + 1\right)$
   (e) $x^6 + x^5 + 5\,x^4 + 4\,x^3 + 8\,x^2 + 4\,x + 4 = \left(x^2+2\right)^2\,\left(x^2 + x + 1\right)$
   (f) $x^4 + x^3 + 3\,x^2 + 2\,x + 2 = \left(x^2+2\right)\,\left(x^2 + x + 1\right)$

2. Let K be a field and $f, g \in K[x]$. Prove that for any $a, b \in K$:

   $$D(af + bg) = aD(f) + bD(g)$$

3. Suppose characteristic of K is 0. Let $f \in K[x]$ be such that $\deg f > 1$. Show $\deg D(f) = (\deg f) - 1$.

4. Assuming the product rule, prove the chain rule (using induction):

   $$D(p(x)^k) = k(p(x))^{k-1}D(p(x))$$

   Solution: This is true for $k = 1$. Assume true for $k < n$. Then

   $$\begin{aligned} D(p(x)^n) =& D(p^{n-1}p) = D(p^{n-1})p + p^{n-1}Dp \\ =&(n-1)p^{n-2}p'p + p^{n-1}p' = np^{n-1}p' \end{aligned}$$

5. Suppose characteristic of K is 0. Let $f \in K[x]$ is

   $$f = p(x)^k q(x)$$

   where $k > 1$ and $\deg p > 0$. Show that $p(x)^{k-1}$ is a factor of $D(f)$.

   Solution: Take the derivative using the product rule:

   $$\begin{aligned} Df =& D(p^k q) = D(p^k)q + p^k D(q) \\ =& kp^{k-1}p'q + p^k q' = p^{k-1}(kp'q + q') \end{aligned}$$

6. Find a non-zero polynomial in $\mathbb{Z}_3[x]$ that has zero has its derivative.

7. Let p be a prime. Characterize all nonconstant polynomials in $\mathbb{Z}_p[x]$ that have derivative 0.