Math 331: Homework Due Sept 30

1. Let R be a ring.

   $r \in R$ is called a *zero divisor* if there is an element $x \in R$, $x \neq 0$ such that $xr = rx = 0$.

   If R has a multiplicative idenity, $r \in R$ is called a *unit* if there is an element $x \in R$, such that $xr = rx = 1$.

   (a) Find all zero divisors and all units in $\mathbb{Z}_{12}$.

   (b) Find all zero divisors and all units in $\mathbb{Z}_{13}$.

   (c) Find all zero divisors and all units in $\mathbb{Z}_{14}$.

   (d) Prove or disprove the following statement: It is possible for an element of $\mathbb{Z}_n$ to be both invertible and a zero divisor.

   Solution: Let R be any ring with identity. Suppose $a \in R$ is a unit. Then, $\exists b \in R$ such that $ab = ba = 1$. Suppose that $\exists c \in R$ such that $ac = 0$. Then,

$$
\begin{aligned}
b \cdot (ac) =& b \cdot 0 = 0 \\
(ba)c =& 0 \\
1 \cdot c =& c = 0
\end{aligned}
$$

   and therefore a can not be a zero divisor.

2. Prove or disprove: Let $x \in \mathbb{Z}_n$ be a unit. The the multiplicative inverse of x is unique.

   Solution: Let R be an ring with identity and suppose $a \in R$ is a unit. Then there exists an inverse $b \in R$: $ab = ba = 1$. Suppose c is also an inverse of a. Then

$$b = 1 \cdot b = (ca)b = c(ab) = c \cdot 1 = c$$

   and $c = b$ and the inverse is unique.

3. Let K be a field (for example $\mathbb{Q}$). Prove that for $f, g \in K[x]$ we have

   (a) $\deg(fg) = \deg(f) + \deg(g)$

   Solution: If $\deg f$ or $\deg g$ is $-\infty$ then the equality clearly holds. Thus, lets assume that $\deg f, \deg g \geq 0$. Without loss of generality, we can assume that $\deg g = m \leq \deg f = n$. Then we can write

$$
\begin{aligned}
f &= \sum_{k=0}^{n} a_k x^k \\
g &= \sum_{k=0}^{m} b_k x^k
\end{aligned}
$$

where $n \leq m$, $a_n \neq 0 \neq b_m$. Then,

$$fg = \sum_{j=0}^{n+m} \left(\sum_{k+l=j} a_k b_l \right) x^j$$

It is clear from the definition that the highest power of x is x^{n+m}. And, the coefficient of x_{n+m} is

$$\sum_{k+l=n+m} a_k b_l = a_n b_m \neq 0$$

Thus, $\deg(fg) = \deg f + \deg g$. (Note: $a_n b_m \neq 0$ since we are in a field, this would not hold true if we were in something like $\mathbb{Z}_6$.)

(b) $\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$

Solution: If $\deg f$ or $\deg g$ is $-\infty$ then the equality clearly holds. Thus, lets assume that $\deg f, \deg g \geq 0$. Without loss of generality, we can assume that $\deg g = m \leq \deg f = n$. Then we can write

$$f = \sum_{k=0}^{n} a_k x^k$$

$$g = \sum_{k=0}^{m} b_k x^k$$

where $n \leq m$, $a_n \neq 0 \neq b_m$. Then, by definition,

$$f + g = \sum_{k=0}^{n} (a_k + b_k) x^k$$

where b_k is defined to be 0 for $k > m$.

Looking at this definition, it is clear that the highest power of k is n and therefore $\deg(f + g) \leq n$.

As a note, it is clear that it would be possible to actually have $\deg(f+g) \neq n$, in the case where terms cancelled.