Math 331: Homework Due Sept 16

1. Let E be the triangle in $\mathbb{R}^2$ with vertices $(1, 0)$, $(-1/2, \sqrt{3}/2)$, $(-1/2, -\sqrt{3}/2)$.
   (a) Verify that E is an equilateral triangle.
   (b) Find all symmetries of E, written as matrices.
   (c) Find all symmetries of E, written as permutations. Write these in both cycle notation and function notation.
   (Ideally, you will have a given symmetry written in all three ways so you can see the different ways of writting the one symmetry.)

2. Let C be the unit cube in $\mathbb{R}^3$. The vertices of C are $(\pm 1, \pm 1, \pm 1)$.
   (a) Write down all the symmetries of this cube using matrices.
   (b) Find all symmetries of C, written as permutations. Write these in both cycle notation and function notation.
   (c) Compare this to the group of symmetries of the unit octehedron with vertices at $\pm e_1, \pm e_2, \pm e_3$. Discuss any differences or similarities.

3. Let T be the tetrahedron in $\mathbb{R}^3$ with vertices at $(1, 1, 1)$, $(-1, -1, 1)$, $(-1, 1, -1)$, $(1, -1, -1)$.
   (a) Verify that this is indeed a regular tetrahedron.
   (b) Write down all symmetries of this tetrahedron using matrices.
   (c) Find all symmetries of T, written as permutations. Write these in both cycle notation and function notation.

4. For each of the following permulations in cycle notation,
   - Write the cycle in "function" notation.
   - If the given cycle is a product of disjoint cycles, write as a product of non-trivial non-disjoint cycles.
   - If the given cycle is not a product of disjoint cycles, write as a product disjoint cycles.

   (a) $(1, 2, 5, 4)(8, 1)$
   (b) $(1, 6, 5, 9, 12)(12, 1)(4, 1)$
   (c) $(1, 2)(2, 3)(3, 4)(4, 5)(5, 6)$
   (d) $(1, 2, 5)(6, 4, 9)(3, 7)$

5. Find the inverse of each permutation in Problem 4.

6. (a) Show that any k-cycle $(a_1, a_2, \ldots, a_k)$ can be written as a product of $(k-1)$ 2-cycles.

   Solution: All you have to do is verify that

$$(a_1, a_2, \ldots, a_k) = (a_1 a_2)(a_2 a_3)(a_3 a_4) \ldots (a_{k-1} a_k)$$

   (b) Show that any permutation can be written as a product of 2-cycles.

   Solution: Let σ be an arbitrary permutation. We know (from class and from the textbook) that σ can be written as a product of disjoint cycles:

$$\sigma = \sigma_1 \cdots \sigma_k$$

   where σ_i are cycles. Then, from the previous problem, each σ_i can be written as a product of 2-cycles. Substituting these in gives σ as a product of 2-cycles.

7. For $n \geq 3$ show that S_n is not abelian.

   Solution: For $n \geq 3$ the two cycles $(12) \in S_n$ and $(23) \in S_n$ but $(12)(23) \neq (23)(12)$. Thus, S_n is not abelian.

8. Let σ be an element in a group (for example S_n). The *order* of σ is the smallest positive integer n such that $\sigma^n = e$. State and prove a theorem about the order of a k-cycle in S_n.

   Solution: Let $\sigma = (a_1, a_2, \ldots a_k)$ be a k-cycle. Then, viewing σ as a function, we can also write

$$\sigma = (a, \sigma(a), \sigma^2(a), \ldots, \sigma^{k-1} a))$$

   Notice that for any $1 < m < k$, we have $\sigma^m(a) \neq a$ but that $\sigma^k(a) = a$. Thus, $m = k$ is the smallest positive integer such that $\sigma^m(a) = a$. Similary, you should be able to see that $\sigma^k(a_i) = a_i$ for any of the numbers inside of the k-cycle σ.